Modal Harmonics and the Coding Patterns of Nature

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Abstract

We know the fundamental constants *e* and π can be utilized to generate solutions explaining real world phenomena but the constants themselves have previously defied attempts at their derivation or explanation as to their origin. We pose that *e* and π are generated by waves harmonically ordered which cannot be observed because they occur in an alternate space. We call this alternate space Harmonic Space. We have discovered an equivalence relation which we term the General Transform Equation that allows us to mathematically transform wave combinations in Harmonic Space to real world phenomena in observable space. The combination is a recipe of which base waves to use and how much of each in order to equate to a specific phenomena such as the measurement of a fundamental constant. The coefficients applied to the base waves necessary for equivalence are in fact a coding pattern which determines the value of these fundamental constants.

A. Introduction

The fundamental constants e and π are considered basic and cannot be derived. Nature, however, is tasked with providing these values and a means of implementation. This mechanism has proven elusive because insufficient variables exist in real space and these fundamental constants are unit-less and transparent.

We describe (a mapping technique involving) an underlying virtual space which is rich in structure and provides the necessary variables to maintain natural order. It is composed of waves harmonically ordered similar to the Fourier Transform.

Harmonic space is married to real space through an equivalence relation which requires that the two observation methods agree on the result of a common event. For each fundamental constant the equivalence relation has a specific solution set. It contains the coding pattern for combining the individual harmonics for equivalence. Each combination establishes a specific harmonic mode. Specific modes correspond to the fundamental constants and determine their value.

This paper derives the fundamental constants e and π from first principles using this equivalence relation and describes the method of implementation. The procedure is expanded to include the Fibonacci Series.

B. Procedure

This is the general procedure. First we set up an equivalence relation called the General Transformation Equation or GTE. In its most general form:

GTE
$$\int_A^B f(x)dx = \int_A^B F(X)dX$$

The left hand side (LHS) of the equation represents an action $A \rightarrow B$ in real space x. The right hand side (RHS) represents the mapping of this same event in harmonic space X. Since they both describe the same event they must agree on the result. Hence the integral of the mapping function in one space must equal the integral of the mapping function in the other space.

The harmonic mapping function $F(X)$ and the boundary conditions A and B must be properly chosen for equivalence. Typically we graph the left hand side in x space and the right hand side in X space to demonstrate they are graphically equivalent over the entire range.

Harmonic space is made up of an infinite series of wave functions. When the right hand side is evaluated at the boundary conditions it will define a specific solution set. Since the two sides are the results of alternative mappings of the same even they must agree on the result. This establishes a relationship between the fundamental constant in real space and the solution set in harmonic space. Not only does it define the constant's value, it also provides the underlying mechanics that require the constant to have this value.

C. Similarity to the Fourier Transform

Our approach is similar to the Fourier transform. In Fourier analysis a periodic signal, which by nature varies in time, is expressed as the sum of component waves. The component waves are numbered and vary in frequency and amplitude. When these individual waves are added together they result in the

original signal. We view this mathematical transform as a remapping of the signal from one space to another (time to frequency). We use the term "space" liberally here to mean any variable we can map our data onto including time and frequency. The importance of this is that a signal which appears highly irregular and mathematically unmanageable in the frame we are accustomed to viewing it, in this case it is time, is more naturally expressed in terms of frequency.

Fourier showed that a periodic function can be expressed exactly, or "transformed", into the sum of a series of sine or cosine wave functions. See Figures 1A & 1

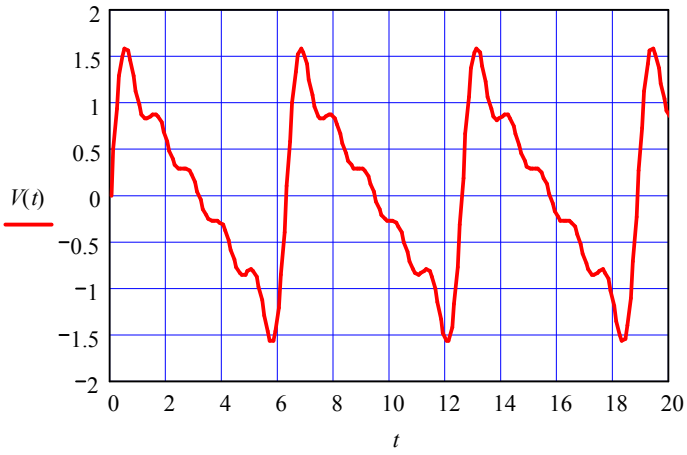


Figure 1A

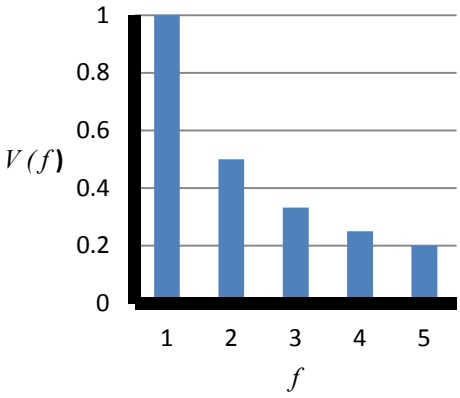


Figure1B

In Figure 1A $V(t)=\sin(t)+\frac{1}{2}\sin(2t)+\frac{1}{3}\sin(3t)+\frac{1}{4}\sin(4t)+\frac{1}{5}\sin(5t)$ is displayed in time space, and in Figure 1B the same function is mapped in frequency space as $V(f)$. We see that $V(t)$ appears to be very erratic when graphed in time space. Yet it is just the sum of five simple waves of progressively finer frequency and smaller amplitude. $V(f)$ indicates the amplitude of each of the five components $V_1(t)$ through $V_5(t)$ that add to equal $V(t)$.

Figure 1B is somewhat analogous to a key. A real key is cut in a specific pattern so that each pin in the lock is raised to a specific elevation to open the lock. In the same way $V(f)$ indicates the amplitude of each component wave that is needed. It is a key that contains the coding pattern needed for equivalence between the mapping spaces.

Figure 1C shows the five component waves and their sum, *V(t)*. Note that the components have progressively smaller amplitude and higher frequency and that adding the component waves produces a function equivalent to *V(t)*. The steep initial rise and spike is due to all the components increasing in this interval. The smaller humps are due to the higher frequency and smaller amplitude components. Lastly, all of the components have a value of zero at multiples of π and therefore *V(t)* must go to zero here as well.

Our approach is similar to this. An unexplainable natural constant like *e* or π is found to model well as the sum of a series of ordered wave forms under certain boundary conditions. A specific amount of each individual component wave is needed for this to occur, analogous to a key. There is a coding pattern that determines all of these amounts. Evaluating the right hand side of the GTE at the boundary conditions determines this pattern. The GTE also demands that the results from the two mapping systems agree. In this way the fundamental constant in real space relates directly to the coding pattern in harmonic space.

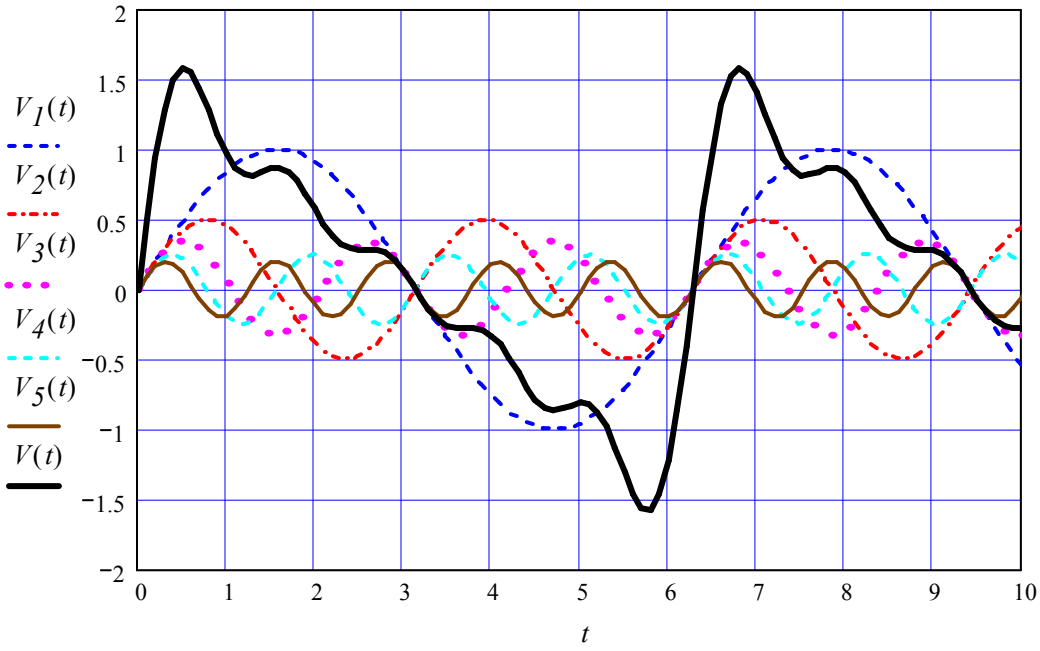


Figure 1C – The five component waves $V_1(t)$ thru $V_5(t)$ add to form $V(t)$

D. The Five Spring Theatre

This analogy provides a physical example of observing the same phenomena from different spaces. We have an audience in a theatre viewing a screen which is projected at from behind with a bouncing light. See Figure 2. The audience sees a light which bounces up and down in an intriguing but erratic way but they cannot see the device which moves it. It is a complex motion and the viewers cannot picture any simple device that would do this. If we were to put this light on cart and roll it across the stage it might make the pattern of a local mountain or building or some other complex design. We would need some sort of translucent and phosphorescent screen so that the part which has already been "drawn" does not disappear. The audience's perspective is that of time space and they cannot view frequency space.

Behind the curtain is a stagehand. He sees the light as a small rod suspended from five springs of different size and frequency that are connected to each other, the largest at the top and the smallest at the bottom. From here it does not look mysterious. Each spring simply bounces at its own natural rhythm. To the stagehand the system moves in a natural way. He doesn't see the complex light movements and cannot appreciate the pattern of a nearby mountain created as the cart and bouncing light roll across the stage. To

him there are five connected springs vibrating in a normal way on a rolling cart and the magic and mystery are gone. The stagehand's perspective is that of frequency space.

This demonstrates how viewers in a different space might view the same phenomena. The audience in the theatre sees a complex motion in time which they cannot explain. The stagehand behind the screen sees the motion as logically based on mechanics and frequencies, but can't see the pattern.

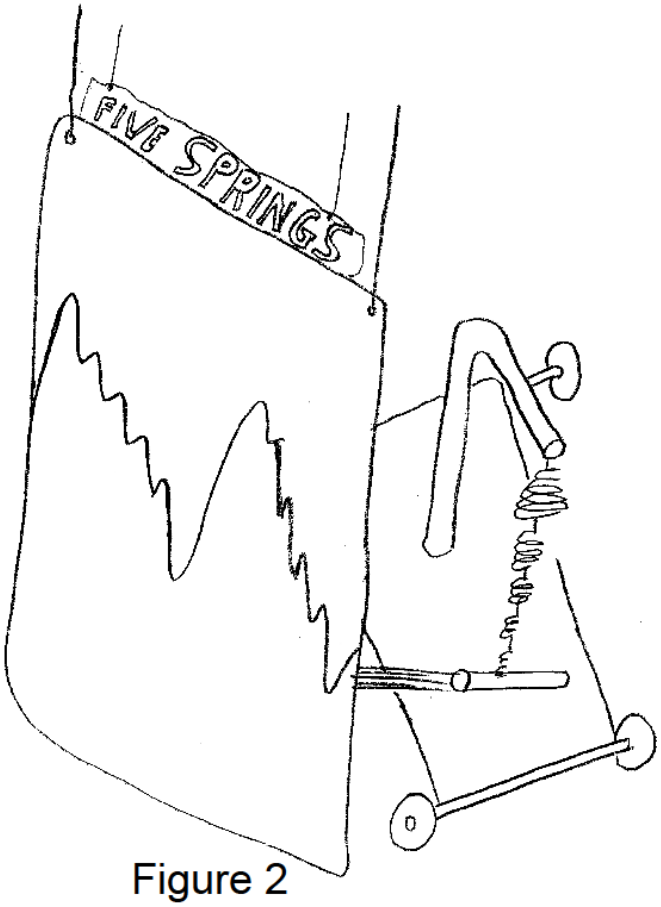


Figure 2

This brings us to Modal Harmonic Analysis or MHA. ***Since both viewers are observing the same physical event, the result that each arrives at from their different mappings must be in agreement for that common physical event.*** In our example the movements of the individual springs on a rolling cart must add up to produce the intriguing pattern the audience sees.

This is not necessarily intuitive despite the simplicity of the words. Equating different spaces is like equating apples and oranges. Does it require us to believe that apples might equal oranges? No. People in apple space must count with apples and those in orange space must use oranges. They just have to agree on what actually happened. Mathematically we would say that the integral within apple space equal must equal the integral within orange space. That is the basis of this analysis.

Part 1: *e*

A. Definition of e^x

We will use the function $f(x) = e^x$ which we define to be the function that is equal to its own derivative. That is to say that if we were to pick any point x on the curve we would find that the corresponding value $f(x)$ is also equal to the rate that $f(x)$ is increasing, or equivalently, the slope of a straight line drawn tangent to the curve at this point. This pattern often occurs in nature. We might expect the growth in insect population to be proportional to the number of insects we currently have, or the dying off or decline of a natural phenomena to also be proportional to its current value.

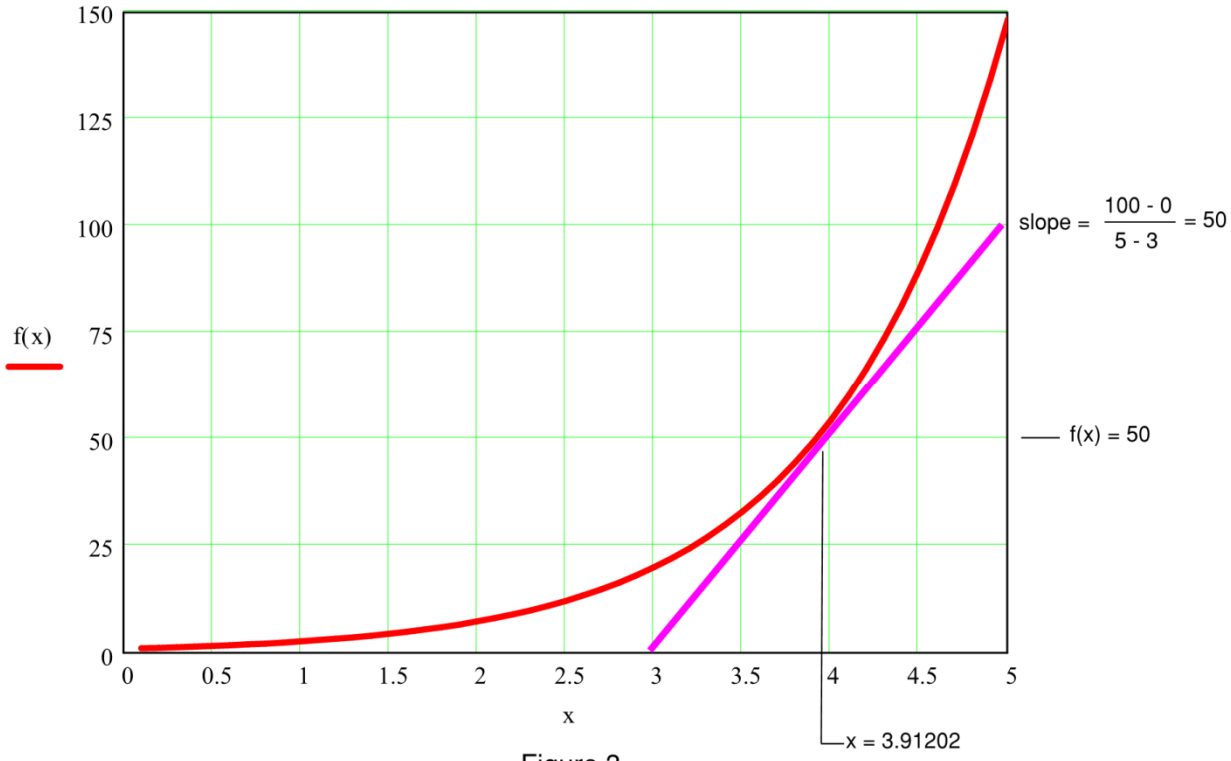


Figure 3

Graphically we depict this in Figure 3. If we draw a line tangent to the curve at any point, the slope of the tangent line represents the rate that curve is increasing. At $x = 3.91202\ldots$, $f(x) = 50$ so the value of the function is 50 at this point. The slope (rise divided by run) of the tangent line also has a value of 50 at this point, so at $x = 3.91202\ldots$, $f'(x) = 50$ as well. This relationship holds for all values of x.

A better mathematical description for the derivative would involve evaluating $f(x)$ at a point slightly less than x and a point slightly greater than x, each differing from x by an amount Δx, and subtracting these values to get the rise. The run would be $2\Delta x$, and so the derivative would be

$$f'(x) = \frac{e^{x+\Delta x} - e^{x-\Delta x}}{2\Delta x}.$$

As noted, the value of $f'(x)$ is equal to $f(x)$ for all x. We can express this as $f'(x) = f(x)$ where $f(x)$ is our function e^x and $f'(x)$ is the derivative. This is our definition of e^x. It is the function that is its own derivative. This differs from the traditional definition of e^x but is mathematically equivalent.

B. Importance & Motivation

Why is the function $f(x) = e^x$ being equal to its rate of change $f'(x)$ so important? First, it is because such relationships occur frequently in nature. For example, the growth rate of an unchecked animal population at any time might be equal (or proportional) to the population. Or, in describing decay, the rate a radioactive material decays would be equal (or proportional) to the amount of material present. Since this function e^x is so important we are naturally inclined to ask what the underlying value of e is. And, therein lies the problem. e is in fact an irrational number, $e = 2.718\ldots$. How on earth can this number which is integral to natural phenomena be so unnatural that it is irrational? Even more perplexing, it can be expressed as the sum of infinite components:

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \ldots \qquad \text{or in summation notation } e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

This is the formula we will derive. The messy irrational number e in real space is a natural outcome of the function being equal to its rate of change when it is mapped in harmonic space. The relation allows a physical event to be described by an alternate set of variables which map the event for another observer in the alternate space. The two different observers must agree on the outcome when they are viewing the same event and in this way the two spaces are related. The harmonic space, however, is rich in complexity and has rules of combination that must be obeyed when describing a physical event in the same way we might have rules describing falling bodies in real space. These rules impact the way the process occurs and are integral to it. Surprisingly the rules in harmonic space can have a physical manifestation in real space, even though we cannot observe the spatial variables themselves. This is because both are views of the same event, and the harmonic space has rules that the process must obey. So although we cannot view harmonic space from real space, there are still signs of it. Most noteworthy is that MHA determines the value of the fundamental constants e and π.

x is the variable we will use in real space and $f(x)$ is the mapping function in real space. Thus $f(x) = e^x$. Harmonic space is based on the variable X raised to the n^{th} power and the n^{th} harmonic has a base of X^n. The n individual harmonics are summed to form $F(x)$. This is the mapping function in harmonic space. A physical event measured in either system must yield the same result.

X is in fact unit-less and has a value of 1 when we attempt to view it in real space. This means it is unobservable in real space since $1^n = 1$ for all n. The value of e cannot be determined by an equation in real space since there are no variables in real space. For that reason the harmonic space cannot be observed.

C. Background & Development

$f(x) = e^x$ is the function in real space (x space) that equals its own derivative:

$$f(x) = e^x \quad \text{and} \quad f'(x) = e^x, \quad \text{therefore } f'(x) = f(x)$$

That is our definition of e^x.

We have an event that occurs in real space from point A to point B. We must find a function *F(X)* in harmonic space (X space) that will create an equivalent mapping for that event. That is $F(X)$ must fulfill the requirement:

GTE $$\int_A^B f(x)dx = \int_A^B F(X)dX \qquad \text{(A)}$$

Note that the left side of the equation occurs in x space while the right side occurs in X space. We use A and B to represent the start and end points of the event, also called the boundary conditions. The equation requires that results obtained in x space must equal that in X space when tallied for the action $A \rightarrow B$. This is the General Transformation Equation (GTE).

Since the integrals are performed in their respective spaces x and X , it may appear that we are adding apples and getting oranges. That is not the case. The integrals in Equation (A) represent the conclusion reached after using either apples or oranges for measurement. As an analogy, rival fans experience and measure a game in vastly different ways, but at the end of the day when each sums their own measurement and experience, both arrive at the same final score. That is what the equation is saying.

f(x) we know equals e^x. This is the left hand side of Equation (A) which occurs in real space. We need to find the function *F(X)* in harmonic space to put into the right side of Equation (A) so that the results of a physical event measured in the two spaces space agree.

To find the *F(X)* that solves Equation (A) we do not need to execute the integrals of that equation in this particular case. We are working with e^x which is the function that is its own derivative. Since integration and differentiation are inverse operations, not only is *f(x)* its own derivative, it is also its own integral:

$$f'(x) = f(x) \quad \text{implies} \quad \int f(x)dx = f(x) \qquad \text{for } f(x) = e^x \text{ only}$$

Since the integral is equal to the underlying function in this particular case, in the GTE we can replace the integral with its underlying function and avoid integration. So in this specific case Equation (A) can be simplified to require that the two underlying functions *f(x)* and *F(X)* must be equal. Therefore the requirement to satisfy Equation (A) can be simplified:

General Transform Equation $$\int_A^B f(x)dx = \int_A^B F(X)dX \qquad \text{(A)}$$

for $f(x) = e^x$ only, implies $$f(x) =_g F(X) \qquad \text{(B)}$$

Equation (B) should be understood to mean that our function *F(X)* mapped in harmonic space must be graphically equivalent to *f(x)* mapped in real space. That means when *F(X)* is mapped in X space the graph obtained is identical to that when *f(x)* is mapped in x space. In that sense they are equal, but that is the end of the similarity. *f(x)* describes a natural phenomena in real space where growth or decay rate is proportional to the quantity present. *F(X)* is an infinite series of waves that add up to give the same result. Clearly Equation (B) is not an equation we could use algebraic operations on, moving terms across the equals sign with algebraic operations. Since Equation (B) is only about equivalence in mapping values we use the sign "$=_g$" to mean graphically equivalent.

Equation (B) also implies that *F(X)* is its own derivative in X space. This must be true since *f(x)* is its own derivative in x space and the two functions are graphically equivalent when mapped in their own spaces. This implies Equation (C):

Implied by (B) $$F'(X) = F(X) \tag{C}$$

This is a differential equation in X space that can be solved. We are looking for the function *F(X)* which is its own derivative. We will try as a solution to Equation (C):

$$F(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n \tag{D}$$

F(X) is an infinite series with each component wave function given a number n. Each wave function is the product of an amplitude factor and a harmonic base of X^n. The amplitude factor for each individual component is $\frac{1}{n!}$.

To demonstrate that (D) is the solution to the differential equation (C) we expand the series of harmonics described by (D) to show the components:

$$F(X) = 1 + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \frac{1}{24}X^4 + \frac{1}{120}X^5 + \ldots \tag{E}$$

Taking the derivative of each component in the series:

$$F'(X) = 0 + 1 + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \frac{1}{24}X^4 + \frac{1}{120}X^5 + \ldots \tag{F}$$

We notice that *F'(X)* does in fact equal *F(X):*

$$1 + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \frac{1}{24}X^4 + \frac{1}{120}X^5 + \ldots \;=\; 0 + 1 + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \frac{1}{24}X^4 + \frac{1}{120}X^5 + \ldots \tag{G}$$

Therefore (D) is in fact the solution to the differential equation (C). If we graph the functions *f(x)* and *F(X)* we can see the graphs are identical (See Figure 4).

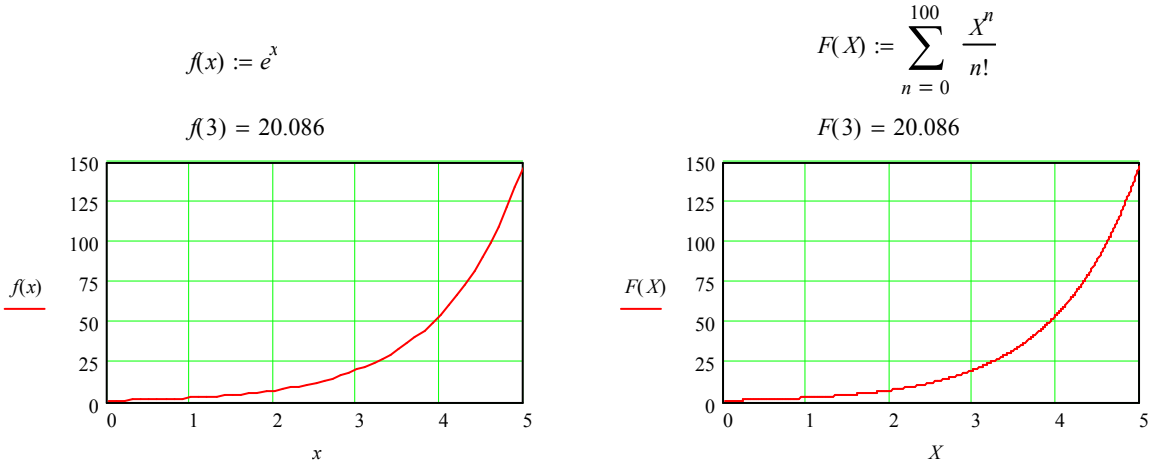


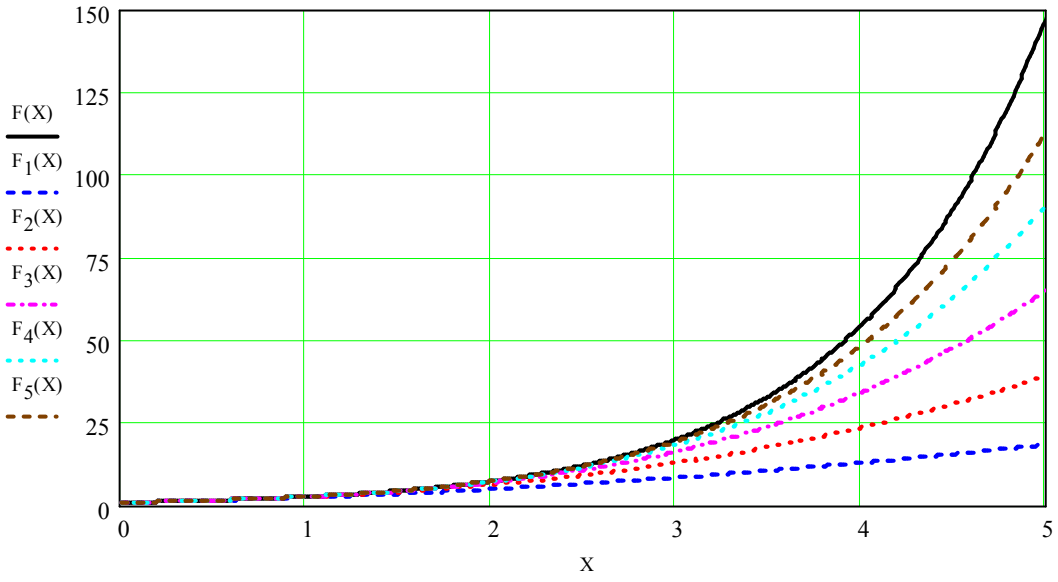
Figure 4 – *f(x)* and *F(X)* are graphically equivalent. This sum of harmonics maps identical to e^x.

This is surprising. What is perhaps the simplest natural function, e^x, can also be defined as an infinite sum of harmonics with each harmonic having an amplitude of $\frac{1}{n!}$.

Figure 5 shows progressive summing of modes 2 though 6 and *F(X)* which is the sum of all of the modes and is equal to *f(x)*. Each additional mode adds a smaller amount resulting in convergence.

Figure 5 – Summation of Modes 2 – 6 illustrates convergance.

$F_1(X) := \sum_{n=0}^{2} \frac{X^n}{n!}$ $F_2(X) := \sum_{n=0}^{3} \frac{X^n}{n!}$ $F_3(X) := \sum_{n=0}^{4} \frac{X^n}{n!}$ $F_4(X) := \sum_{n=0}^{5} \frac{X^n}{n!}$ $F_5(X) := \sum_{n=0}^{6} \frac{X^n}{n!}$



We call this modal harmonics since each of the n modes that X undertakes (X^0, X^1, X^2, X^3... X^n) is of greater order and smaller amplitude $(1,1,\frac{1}{2},\frac{1}{6},\frac{1}{24},\frac{1}{120})$. See Equation (E) and Figure 6.

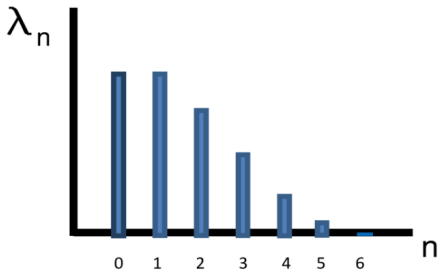


Figure 6

As an analogy we look to Fourier's transformation of a signal in time as being the sum of progressively finer wave modes each with its own amplitude coefficient.

To summarize *F'(X) = F(X)* is the differential equation (C) with solution $F(X)=\sum_{n=0}^{\infty}\frac{1}{n!}X^n$ (D) that the GTE requires for physical actions involving *e*. This solution can be seen to be the sum of wave forms, each made up of two factors two factors: $\frac{1}{n!}$ and X^n. The harmonic base is given by X^n and gives the wave shape. The factor $\frac{1}{n!}$ is the amplitude factor which determines how large a contribution that particular harmonic makes. We rewrite the amplitude factor as λ_n where $\lambda_n=\frac{1}{n!}$.

The solution to the Equation (C) DE can therefore be expressed as:

$$F(X)=\sum_{n=0}^{\infty}\lambda_n X^n \quad \text{with} \quad \lambda_n=\frac{1}{n!} \text{ (amplitude factor)} \qquad \text{(H)}$$

This infinite sum of this harmonics series produces a function *F(X)* in harmonic space that is graphically equal to its counterpart *f(x)* in real space. However the individual harmonics must combine in a specific way for this to happen. The requirement is that

$$\lambda_n=\frac{1}{n!} \qquad \text{(I)}$$

We observe from Equations (E), (F) and (G) that an amplitude factor of $\frac{1}{n!}$ applied to the nth harmonic is the key to having the derivative equal to the function in harmonic space, or *F'(X)=F(X)*.

λ_n must have this value. If it did not then when we sum the $\lambda_n X^n$ we would not get a function that is equal to its own derivative. Then *F(X)* would not be graphically equivalent to *f(x)*. This would violate the GTE (A) and is forbidden.

D. Modal Harmonic Equations for Determination of the Fundamental Constant *e*

Let $f(x) = e^x$. This is the equation in real space we seek the transformation of. *f(x)* is defined as the function that equals its own derivative, thus $f'(x) = f(x)$. Let *F(X)* be the transformed function in harmonic space that we seek. According to the GTE it is required that when we measure a physical phenomenon we get the same result regardless of which space is used:

General Transform Equation $$\int_A^B f(x)dx = \int_A^B F(X)dX \qquad (1)$$

This is the General Transformation Equation (GTE). However, in this special case where $f(x) = e^x$, the derivative of the function has the same value as the function, or $f'(x) = f(x)$. This implies that the integral of *f(x)* also have the same value of *f(x)* since differentiation and integration are inverse operations. That is the left hand side of Equation (1) operating in x space. In order for Equation (1) to be true for the action between any two points A and B common to both spaces, *F(X)* must be *graphically equivalent* to *f(x)* and we can relate the underlying functions of the GTE without evaluating the GTE integrals. So for $f(x) = e^x$ only Equation (1) implies:

$$f(x) =_g F(X) \qquad (2)$$

The symbol $=_g$ means graphically equivalent. By this we mean they have the same numerical result when given the same input. We cannot, however, do algebraic operations across the equals sign because the left and right sides are of different spaces.

F(X) must also be its own derivative since *f(x)* is its own derivative and they are graphically the same:

$$F'(X) = F(X) \qquad (3)$$

Equation (3) is a differential equation (DE) in harmonic space. The solution is:

$$F(X) = \sum_{n=0}^{\infty} \lambda_n X^n \quad \text{with} \quad \lambda_n = \frac{1}{n!} \text{ (amplitude factor)} \qquad (4)$$

This represents the sum of n harmonic modes (X^0, X^1, X^2, etc.) with an amplitude factor λ_n for each. When summed this function in X space graphs identically to *f(x)* in x space even though it is composed of individual wave functions. So long as we choose the same input value for *x* and *X* we will get the same results from *f(x)* and *F(X)*.

Returning to Equation (2): $f(x) =_g F(X)$, we substitute using $f(x) = e^x$ and $F(X) = \sum_{n=0}^{\infty} \lambda_n X^n$:

$$e^x =_g \sum_{n=0}^{\infty} \lambda_n X^n \qquad (5)$$

We can choose any value of x and X for evaluation so long as we give both x and X the same value.

Since we are looking for the value of *e* and not a power of it, let $x = 1$ and $X = 1$ in Equation (5). Then $X^n = 1$ for all n and the harmonic component will drop out:

$$e^1 = \sum_{n=0}^{\infty} \lambda_n \cdot 1^n \quad \Rightarrow \quad e = \sum_{n=0}^{\infty} \lambda_n \tag{6}$$

e must have this value since Equation (5) was a substitution into Equation (2), which is a form of the GTE for this particular case. Thus MHA requires that the value of *e* in real space be the sum of the eigenvalues that form the solution to the Equation (3) DE in harmonic space.

Substituting $\lambda_n = \frac{1}{n!}$ from (4) into (6):

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \tag{7}$$

This is the formula we set out to derive. *e* must have this value or the GTE would be violated.

We have returned to the standard equals symbol in (6) and (7) because the harmonic has dropped out and algebraic operations are permitted. Note that by selecting $x = 1$ and $X = 1$ for evaluating (5) the harmonic space disappears from the solution. This is not trivial. Solving the equation requires that harmonic space be unobservable from real space. Thus in the transition from (5) to (6) we go from graphically equivalent to equal. But the solution also requires that traces of the harmonic space be directly observable, namely in setting the value of *e*. *e* must equal the sum of the modal amplitudes λ_n which are the eigenvalues that solve the differential equation $F'(X) = F(X)$ in harmonic space.

Since either space could be used to map the same event, physics requires that they both produce the same result when evaluated over the same action. This is the meaning of the GTE. Harmonic space is richer and more complex than real space and only a specific solution set will cause the harmonics to add up to create the function that is its own derivative and solve the DE (3) which the GTE requires. ***Therefore the harmonic space requires that the physical phenomena occur in a certain way and there are specific rules of combination. Additionally the mechanics behind the action cannot be viewed in real space, similar to what we saw in the Spring Theatre example.***

At deeper level we could say that *e* must have a value assigned to it for natural phenomena to operate, but there are insufficient variables in real space to do this. Tasked with this problem nature relies on a virtual space to control and access this value. The counterpart of e^x in harmonic space is a DE which is solved only when the harmonics add up in an exact way. This can be likened to a lock with *n* pins with the key contained in the solution to the DE. The requirement is that the n^{th} harmonic must have an amplitude factor of $1/n!$:

$$\lambda_n = \frac{1}{n!} \tag{8}$$

This is the method that nature uses for storing and accessing the fundamental constant *e*. Any time that a phenomena occurs that utilizes *e* it invokes this DE in the same way that plucking a guitar string would invoke that DE. The solution to the DE contains the coding pattern as the amplitude factors for the individual harmonics, the DE is the method of storing the pattern in a timeless and indestructible way, and the GTE is the method of accessing and executing it.

Part II. A. Debonking the Euler Equation $e^{i\pi} + 1 = 0$

In this section we debonk the dreaded Euler Equation. See Reference (1). It has driven many bonkers.

My brother Barry is a balloonist. In fact, he is the best balloonist in the world. He was born in a balloon and has spent his whole life there.

One day he called me on his cell phone. "There is something very peculiar going on here," he said. "I am watching a ball floating down the river. It goes fast but then slows down and stops, then speeds up again. The pattern keeps repeating."

I went down to the river to experience this strange phenomenon and this is what I saw. The river was wavy, and when the ball was at the top or bottom of the wave it was moving purely in a horizontal direction, but between these points some of the motion was up or down. And midway between top and bottom there was no horizontal motion at all, only vertical, making it appear motionless from the balloon.

This is illustrated in Figure 10. At positions 1, 4 and 6 the motion is completely horizontal: downstream. At position 3 the ball is moving downwards but not forwards. At position 5 it is moving upwards but not forwards. Position 2 has components in both the vertical and horizontal directions.

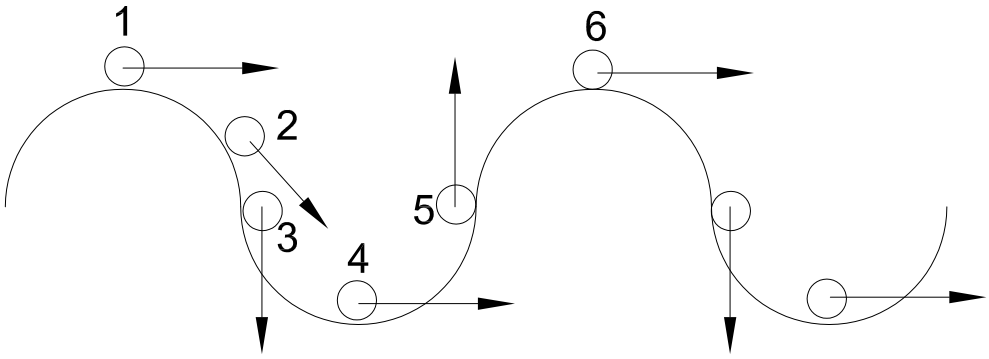


Figure 10 - Ball in Wave

From his balloon Barry could not see the waves or the vertical motion. He could only see the horizontal component of the velocity which cycled between a maximum and zero. I told him he needs to imagine the waves and the up and down motion they create. He was OK with that but wanted a mathematical way to describe it. This is what we came up with.

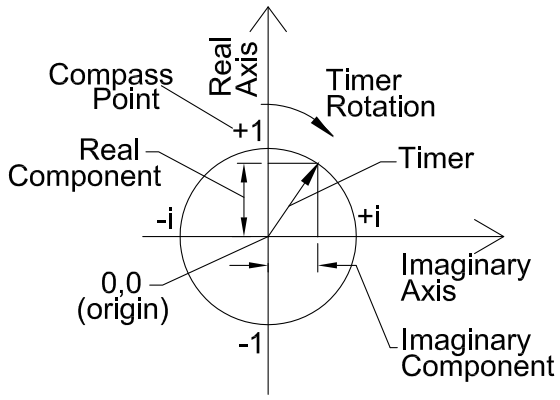


Figure 11A - The timer position indicates the real and imaginary components

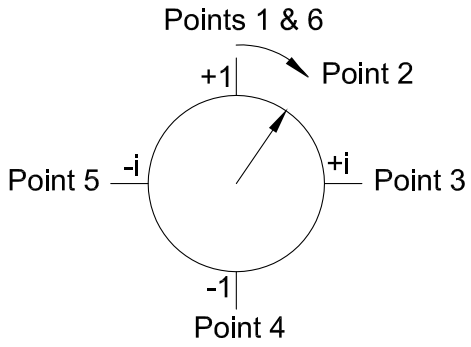


Figure 11B - Timer Position to wave Position in Figure 10

There was a dimension that Barry could not see, and the motion was dividing itself between the part he could see and the part he couldn't see. We set up a timer that spins in sync with the wave as shown in Figure 11. When it pointed perfectly up or down, all of the motion would be visible (horizontal). When it pointed exactly right or left, all of the motion would be in a direction he had to imagine (vertical), which we designated $+i$ and $-i$ on our timer. And between these the motion would be distributed amongst the two directions. Our timer, a one dimensional device, could tell us about motion in two dimensions. It is able to do this because of constraints. There is only so much water and it has to divide itself between vertical and horizontal motion. The vertical and horizontal motions are not independent, if we know one we can calculate the other.

The $+i$ and $-i$ positions indicate that all of the motion is in the imaginary direction, the one he could not see, and the +1 and -1 positions indicate that all of the motion is in the visible (real) direction. In between these four spots the velocity has both an imaginary and a real component.

The Compass Model

Thus far we have treated $+i$, $-i$ and +1,-1 as compass points. We have not assigned numerical values to them, they simply occur at the quarter points of the circle. These quarter points could be determined with simple geometric tools like a compass. In this type of model the closed circle is the base unit. We talk about half circles and quarter circles or cycles which are in fact multiples or fractions of the circle. When we use MHA to determine the value of π we will use this system. However for the moment we will diverge and look at the numerical properties of i the imaginary number that are used in math.

Imaginary and Complex Numbers

i is defined mathematically as the square root of -1. This is called an imaginary number and is frequently used to describe a second property that may be hidden. A complex number has a real part, A, and an imaginary part, B, and is of the form $A + Bi$. Nature uses these to control two different values with just one complex number.

In our example when the spinner points to +1 the real component A equals 1 and the imaginary component B equals zero. *Note that we always work with a circle of radius 1.* As the spinner rotates the real component decreases and the imaginary one increases. At $+i$ the imaginary component has a value of 1 and the real part is zero. The real and imaginary components represent the horizontal and vertical components of the velocity of the ball in the waves. Thus the operation of a single spinner controls two different properties that are related but have different values.

Raising to an Imaginary Power and the Formation of Spinners

When we raise any number to an imaginary power we create a spinner (or timer) of radius 1. This means that when evaluated it will yield a complex number $A + Bi$ which always has a magnitude of 1. As the pointer turns the values of A and B will change, but they will always have a resultant of 1. When it points up the value will be $1 + 0i$ which equals +1. When it points to the right the value will be $0 + 1i$ which equals $+1i$. When the spinner is somewhere between two compass points there is both a horizontal component and a vertical component. Starting at any value, after one cycle the spinner will return to that value. We show this in Figure 12. Here we raise some number x to the $i\theta$ power.

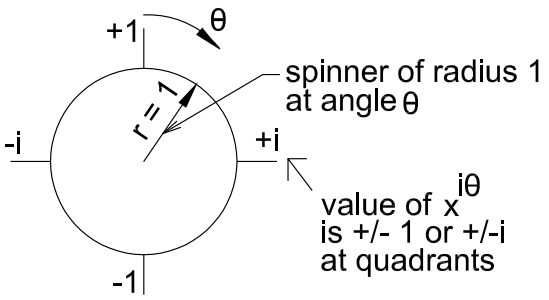


Figure 12A - Raising any number to the iθ power creates a spinner of radius 1. At the four quadrants there is only one component: +1, +i, -1, or -i.

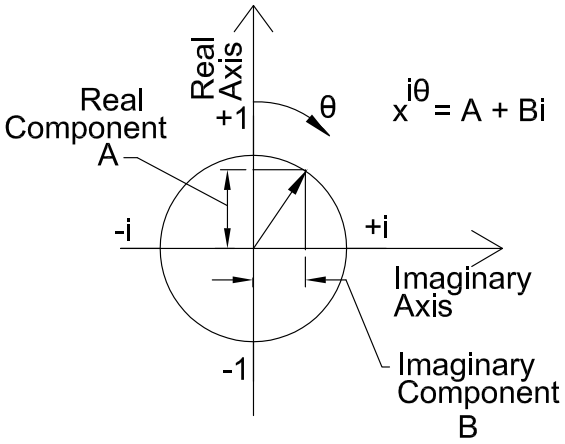


Figure 12B - When the spinner is between the two compass points there is a real and imaginary component

This spinner described by $x^{j\theta}$ models our timer in terms of resulting value. It describes the real and imaginary component velocities as cyclical and out of phase by half a cycle, meaning one is a maximum when the other is at minimum. However it is not calibrated in frequency. It is not in sync with the wave. How do we fix this?

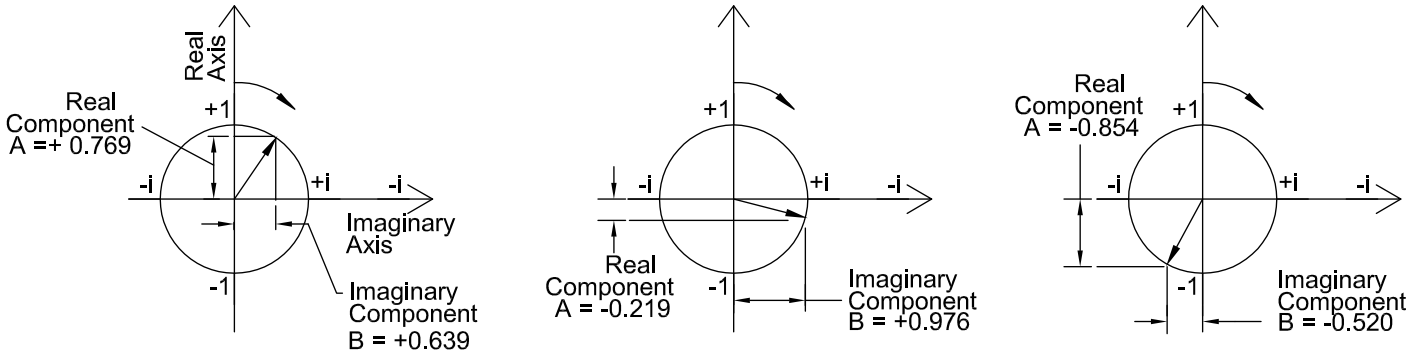


Figure 13 - The higher the base number the faster the spinner turns, but the magnitude is always 1.

Depending on our choice for the base number x, the spinner spins faster or slower, but the magnitude is always 1. In Figure 13 we show spinners at 2^i, 6^i, and 40^i to illustrate how higher base translates to a faster spinner.

What is the right number to use for the base? We will look at $e^{j\theta}$. That is because e^x is that special function that equals its own derivative. It is naturally occurring and is frequently used in the solution of differential equations.

So now we ask, at what angle θ will $e^{j\theta}$ close perfectly? That is, if we start at $\theta_0 = 0$, which is the +1 or North position, at what angle θ does it need to spin to return to +1 (North) again, to close and form a perfect circle?

That special angle is 2π. Every time the spinner turns an angle of 2π it returns to the starting position. **Thus 2π is the angle θ turned that causes $e^{i\theta}$ to close.**

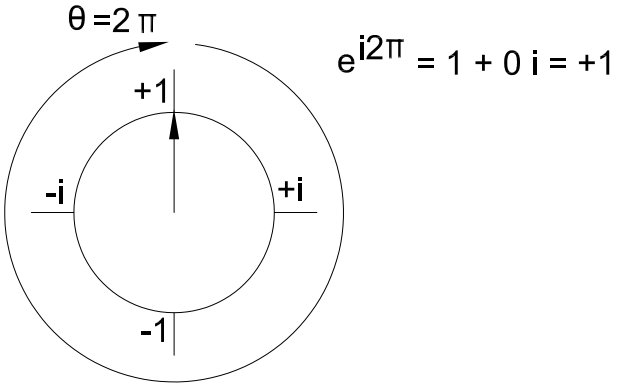


Figure 14 - 2 p is the angle θ that causes $e^{i\theta}$ to close.

Note that we always use an action such as a turn. We did this in our formulation of e but more subtly through the boundary conditions. One reason the fundamental constants have proven so difficult is that they have been studied as static.

π – The Ratio of Lengths or an Angle?

π is usually defined as the ratio of two lengths, the circumference to the diameter of a circle. **Instead we will define π in terms of the angle which we need to turn to achieve closure. Thus we define 2π as the angle θ turned that causes $e^{i\theta}$ to close.**

We consider the circle and more specifically its arc length at closure as a base unit. 2π is a full turn. So π is the angle one must turn to go half way around. This definition of π is fundamentally different from the traditional one. The ratio of circumference to diameter still holds true, and that conversion still applies since r = 1. However instead of working with two quantities (circumference and diameter) we only use one (angle turned).

The Euler Equation : $e^{i\pi}+1=0$

2π is the angle required to make a full turn around a circle, so: 1) π is the angle one must turn to go half way around , i.e. North to South. If we rearrange the Euler equation written above, by subtracting 1 from each side of the equation we get : 2) $e^{i\pi}=-1$. These two underlined statements 1) and 2) are identical in meaning.

We know that $e^{i\pi}$ represents a rotation of an angle π or half way around. What is the significance of this being equal to -1? If we start at North (+1) we will end up at South (-1). If we start at East (+*i*) we will end up at West (-*i*). In general, an arrow from the center to any point on the circle given by A + B*i*, would by be subject to a factor of -1 under this rotation. This causes the arrow to point in the opposite direction. This arrow is of length 1 since it goes from center to some point on the radius and r = 1. It is called a unit vector. Then A + B*i* $\Rightarrow$ - A - B*i* .

Thus the Euler Equation is a description of turning halfway around a circle and states that this will result in the application of a factor of -1 to the original unit vector. If we start at NNE we end up at SSW. The Euler equation is anything but mysterious – it only states the obvious. *If you turn around you will be looking in the opposite direction*. See Figure 15.

By extension a full turn or 2π would result in a factor of +1. In other words we would end up where we started from. This is the same as our definition that 2π is the angle turned that causes closure. The Euler Equation implies our definition and vica versa.

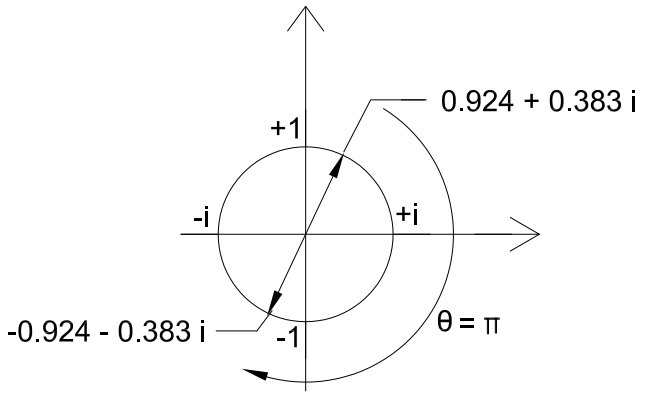


Figure 15 - The Euler Equation is another way of saying that if you turn around you will be looking in the other direction.

More on Compass Points

It is logical to refer to $e^{i\theta}$ at the quarter points of the circle as compass points as this reflects the underlying property of the circle as a base unit. The compass points we have been using are a natural consequence of our definition for 2π as the angle turned for closure. At +1 or the North marker, $\theta = 0$ and $e^{i\theta} = +1$. That is why we identified this point as +1. At +i or the East Marker $\theta = \pi/2$ and $e^{i\theta} = +i$. At -1 or the South Marker $\theta = \pi$ and $e^{i\theta} = -1$. At -i or the East Marker $\theta = 3\pi/2$ and $e^{i\theta} = -i$.

These quadrants of the circle line up exactly on the Cartesian axes so that one of the components is always zero and the other is always 1. So either A = 0 with B = +/- 1 or B = 0 with A = +/- 1. See Figure 16. Since we have divided our full circle of angle 2π into four Cartesian quadrants each must be angle π/2.

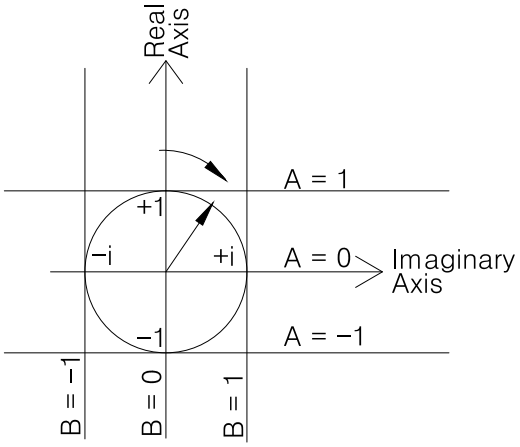


Figure 16 - At the quadrants the A & B values of the complex number A + Bi can be determined by the values of the cartesian plane.

Thus in our π formulation when we recognize +1,+*i*, -1,-*i* as the values for $e^{i\theta}$ at $\theta = 0,\pi/2, \pi, 3\pi/2$, one must realize this is not a backdoor way to introduce a numerical value for π into our equation. The compass point values are a natural consequence of the definition and how the polar equation maps onto the Cartesian coordinate system.

Common Compass Positions We Will Use

If we look at the four values +1, -1, +i, and $-i$ as points on the compass, we see that if we start from one of these points, every time we turn an angle of 2π we return to the starting point:

Starting at $\theta = 0$ the spinner is at +1 (North), and every turn of the spinner by an angle of 2π it will return to this spot, so:

$e^{i0} = +1, \quad e^{i2\pi} = +1, \quad e^{i4\pi} = +1, \quad e^{i6\pi} = +1$

By extension we can see that at $\theta = \pi$ the spinner would point at -1 (South) and would return here again every time an angle of 2π is turned:

$e^{i\pi} = -1, \quad e^{i3\pi} = -1, \quad e^{i5\pi} = -1, \quad e^{i7\pi} = -1$

At $\theta = \pi/2$ the spinner would point at +i (East) and would return here again every time an angle of 2π is turned:

$e^{i\frac{\pi}{2}} = +i, \quad e^{i\frac{5\pi}{2}} = +i, \quad e^{i\frac{9\pi}{2}} = +i, \quad e^{i\frac{13\pi}{2}} = +i$

At $\theta = 3\pi/2$ the spinner would point at -i (West) and would return here again every time an angle of 2π is turned:

$e^{i\frac{3\pi}{2}} = -i, \quad e^{i\frac{7\pi}{2}} = -i, \quad e^{i\frac{11\pi}{2}} = -i, \quad e^{i\frac{15\pi}{2}} = -i$

Part III: Pi 1. Introduction

In this section we show how the value of π can be determined using the General Transformation Equation or GTE. The GTE is the same general formula used for determining the value of e however here we will use a polar coordinate system instead of Cartesian.

Like e, π is a fundamental constant of nature which has an irrational value and can be expressed as an infinite sum. We will derive the famous Gregory Series for π:

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \qquad \text{(III.iA)}$$

Additionally we show the mechanics of how the equation works through a model of two opposing spinners which produce the natural code.

We must have a physical action. Suppose one takes a measurement around the outside of a pipe with a measuring tape. As the tape is laid down more of the circumference is measured, first one inch, then two, then three, etc. Although this occurs on a surface it can also be seen as a one dimensional operation because we can mark the tape, then unroll it, and view our measurements in flat space on a line. Not only is it one dimensional, it has no richness of structure. The arc length is proportional to the angle subtended. With $r = 1$ the arc length and angle are equal. See Figure 20.

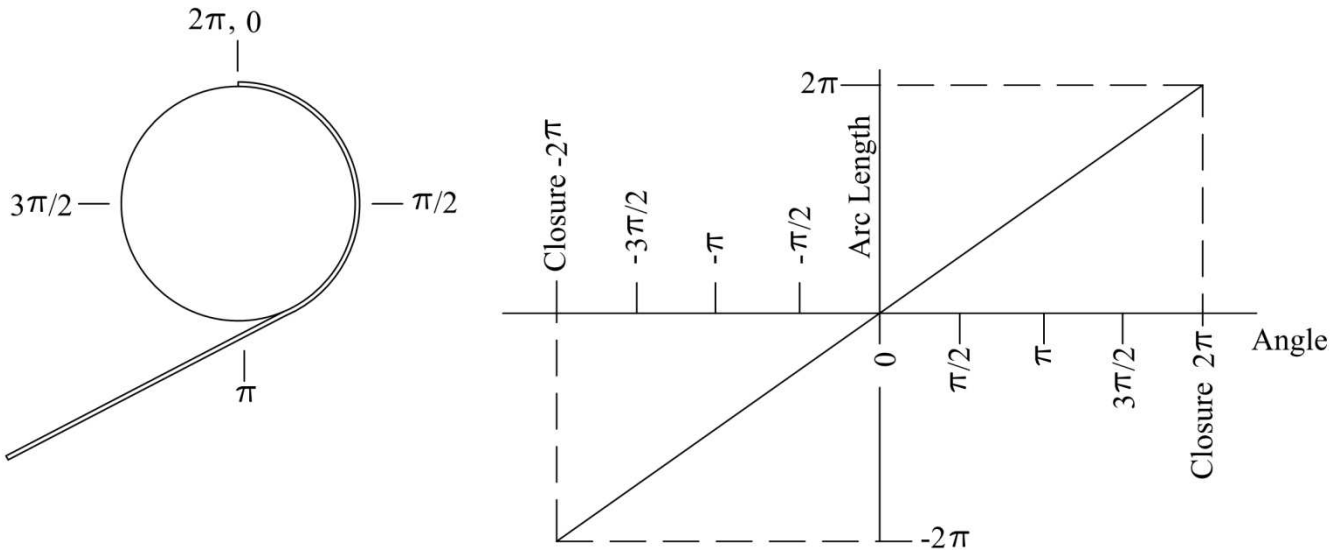


Figure 20 – At left is the measurement of a pipe with a measuring tape. At right the tape is unrolled and the arc length is graphed vs. the angle measured. The surface is two dimensional but the arc length exists only along the circumference and can be mapped in C Space as shown. It is also possible to measure in the other direction and obtain a negative value. The harmonic space model must replicate this graph.

In the Spring Theatre we had a mechanical system behind a screen that had rich structure but the audience could not see it. They could only see the light attached to the mechanism and could not explain its behavior. In this section we will create an equation which graphs identical to the circumferential measurement operation

of Figure 20. The measurement operation occurs in real space. The one we create is mapped in a virtual space composed of harmonic elements which are unit-less and cannot be observed. The reason why we do this is that the harmonic space is rich in structure and because of this there are rules governing the mechanics.

The rules require that the harmonics be combined in a specific pattern in order to equate with the measurement operation in real space. That pattern determines the value of π.

The reason for this is that the equation in harmonic space models the real space equation exactly. They both map a common action but on different types of maps. Since this is a real action there is an equivalence relation that requires that the results observed in both spaces must agree. This equivalence relation provides a numerical relationship between the two spatial maps and they are said to be transformations of one another. The equation for relating these two maps for a single action common to both spaces is described by the General Transformation Equation or GTE. The left hand side represents the action $A \rightarrow B$ in real space and the right hand side represents mapping the same action in harmonic space:

GTE
$$\int_A^B f(\theta)d\theta = \int_A^B F(\Theta)d\Theta \qquad \text{(III.iB)}$$

The two functions $\int f(\theta)d\theta$ in real space and $\int F(\Theta)d\Theta$ in harmonic space are connected. When the measurement operation occurs in real space the rule operations and subsequent mechanics are operating in harmonic space. They are inseparable, like two machines that are on the same shaft and must run together, but are located in different rooms. If one machine is geared to move in increments, both machines must move this way since they are on the same shaft. So when $f(\theta)$ moves an angle $d\theta$ in real space this corresponds to $F(\Theta)$ moving by the same amount when measured relative to $d\Theta$ in harmonic space.

For equivalence, the individual harmonic wave functions must combine in a specific way to form the function $\int F(\Theta)d\Theta$ which maps identical to $\int f(\theta)d\theta$ in real space. Since both are observations of the same event equivalence must exist. The specific harmonic wave combination is a code that invokes rules about how the action can occur. Like the two machines on the same shaft, the two mapping functions are connected and the rules governing actions in harmonic space are effect the actions in real space. In this way the remnants of the harmonic combination rules are detectable in real space. These combination rules determine the value of π. π must have this value. Otherwise the GTE would be violated and the two maps describing the same action would show different results.

Virtual But Describable

We can't see harmonic space but we can describe it mathematically. We can develop equations and rules on paper and see how they work even though we can't observe the space itself.

Harmonic space is wavy and the base units are wave functions or harmonics. The wave functions are numbered and get smaller in amplitude but increase in frequency over the set. There are an infinite number of these harmonic bases available, but since they decrease in amplitude the effect of using additional terms diminishes as the harmonic number n increases. So as $n \rightarrow \infty$, $\int F(\Theta)d\Theta \rightarrow \int f(\theta)d\theta$.

In order to produce the specific function there is a rule set or coding pattern which determines which waves are added, subtracted, or not used. We will see that the rule set for our measurement requires that

$$\frac{\pi}{4} = 1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \frac{1}{7} + 0 - \frac{1}{9} + 0 + \ldots \qquad \text{(III.iC)}$$

or in summation notation

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \qquad \text{(III.iD)}$$

This is the formula we will derive. It is the famous Gregory Series for π which has always been mysterious. The equivalence relation requires that the results of the mappings in the two spaces agree. The consequence of this is that π must have this value.

It is astounding that the component waves can add up to model this fundamental constant. It seems to suggest that harmonic space is more natural for mapping π than real space. Even more astounding is that there is a specific pattern for adding the components and this pattern is encoded in the GTE.

Equally amazing is the way the equation works. When we calculate the left side of the GTE, $\int f(\theta)d\theta$ over the boundary values, we will get a multiple of π. When we calculate the right side, $\int F(\Theta)d\Theta$, all vestiges

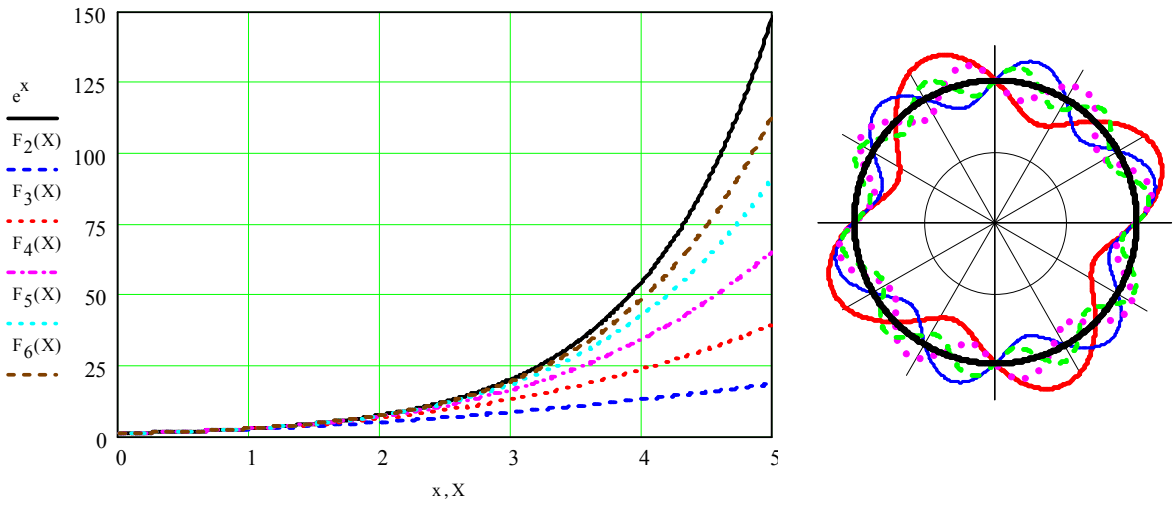


Figure 21 – (Left): In the analysis for *e* we added the wave forms to create a function that models e. The traces shows progressive summing of modes 2 thru 6 and the sum of all modes equals *e*. (Right): A model for visualizing the harmonic modes that sum to form $F(\Theta)$ in the analysis of π. Four of the harmonic components are shown. The harmonic modes are in some ways analogous to vibration modes of a ring.

of the harmonic space disappear and only the rules of how to combine the base waves remain, but not the waves themselves. This establishes an equation relating π and the combination rules in harmonic space.

Thus the transformation equation requires that harmonic space be unobservable and that π be determined by the harmonic combination rules. We saw this occur with *e.* The base waves dropped out of the equation requiring *e* to be equal to the sum of the eigenvalues of the differential equation. The sum of harmonics is used to model both *e* and π (See Figure 21). However in the case of π the harmonics must add up to agree with the results of the measurement taken along the circumference of a circle. We can imagine the harmonic modes as perturbations or vibration modes of a ring, which must add up to form the ring and must exhibit properties of continuity and closure.

In our analysis we will always take the radius of the circle to be $r = 1$. We consider the circle as a base unit and 2π is the angle turned which results in closure. With $r = 1$ the arc length C that is measured has the same value as the angle turned, θ. This is because $C = \int r \cdot d\theta = \theta$. To distinguish between the arc length C and angle turned θ, we will refer to arc length in C space and angle in θ space. See Figure 22 *top left*.

With $r = 1$ we can usually use arc length and angle interchangeably since they have the same numerical value. However it is important to recognize that arc length exists in C space. It is a length measurement confined to the circumference of a circle and is a practical way to take physical measurements. We can unroll this into a Cartesian graph and the function is then mapped onto a straight line (Figure 20). When we use the GTE it is more correct to use θ space or angle turned because harmonic space is unit-less while C space would have some length unit attached. We could overcome this dilemma by referring to the C / r ratio however we feel it is simpler to work with the one number θ.

Figure 22 *top right* is a drawing to help visualize the harmonic modes. They are analogous to vibration modes of ring. These waves are thought of as perturbations of the circumference and they have a special relationship to it. The correct combination of these waves of the circumference sums to equal the circumference.

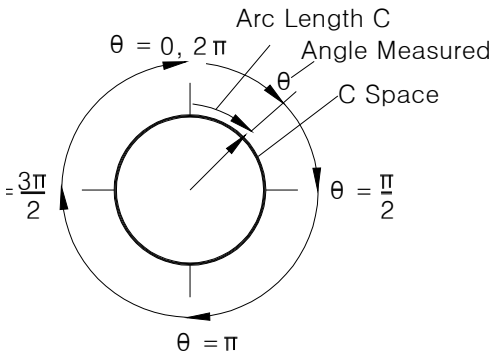
Figure 22 *bottom* graphs the first four harmonics. Note the increasing frequency and decreasing amplitude. The sum of the first four harmonics and the first 100 harmonics are also shown. Note how the sum of the harmonics is trending toward the real measurement; that is the linear graph shown in Figure 20. The relationship gets better with increasing n and is exact when $n = \infty$.

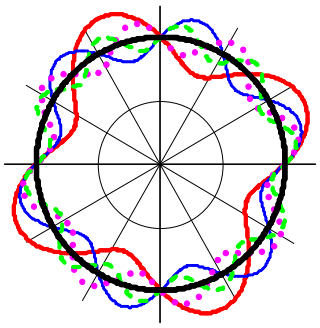
The integrals F_n for the first four harmonics on the graph are based on the formula:

$$\int F_n(\Theta) d\Theta = \frac{2}{n \cdot i} \cdot \left[e^{in\left(-\frac{\theta_m}{2}+\pi\right)} - e^{in\left(\frac{\theta_m}{2}+\pi\right)} \right] \qquad \text{(III.iE)}$$

The summation of the first four harmonics is below. The 100 summation is similar.

$$SumF_4 = \sum_{n=1}^{4} \int F_n(\Theta) d\Theta = \sum_{n=1}^{4} \frac{2}{n \cdot i} \cdot \left[e^{in\left(-\frac{\theta_m}{2}+\pi\right)} - e^{in\left(\frac{\theta_m}{2}+\pi\right)} \right] \qquad \text{(III.iF)}$$





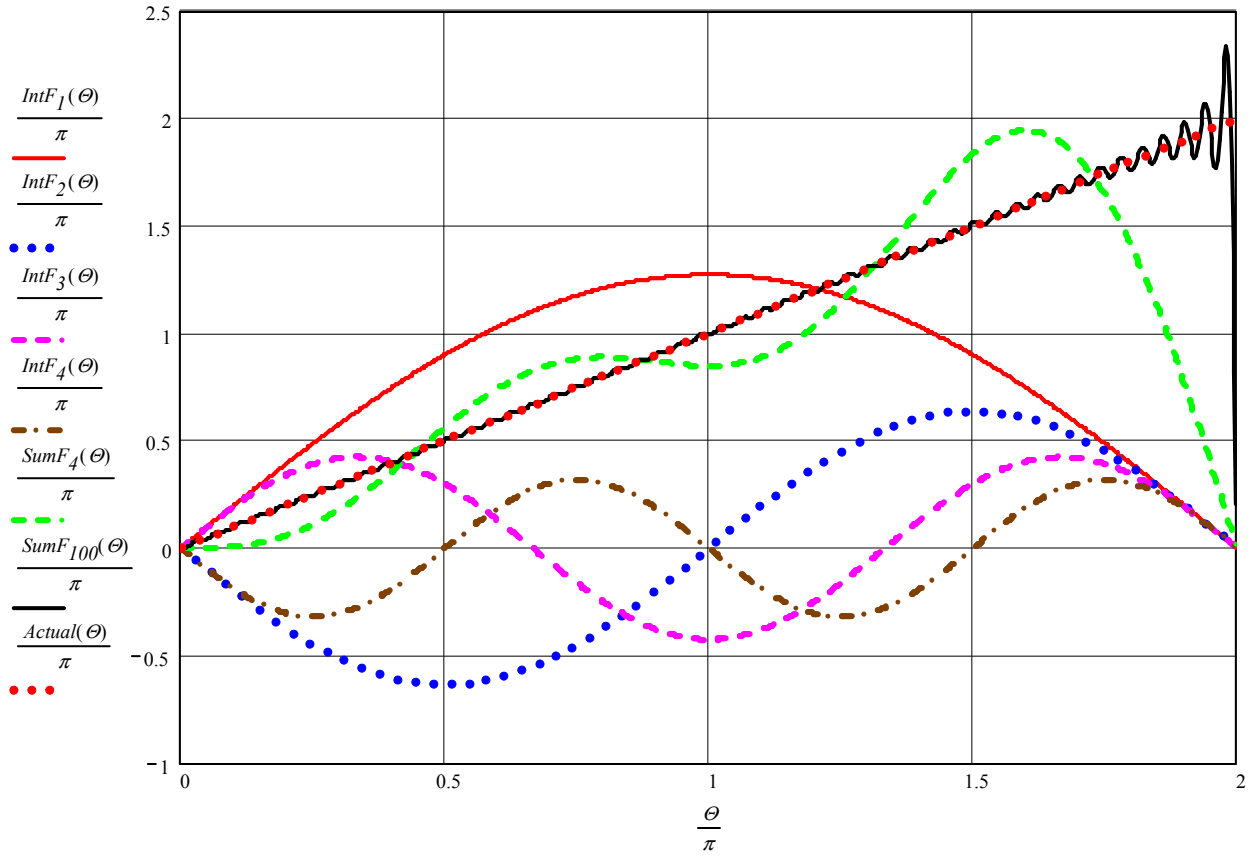


Figure 22 – *Top Left*: the arc length C is confined to the circumference and equals the angle turned θ when $r = 1$. *Top Right*: Some of the harmonic modes. *Bottom*: The circle is unrolled into a flat line with the first four harmonics plotted. The sum of the first 4 and first 100 harmonics is also shown. Note how at $n = 100$ the sum of the harmonics is trending toward the linear graph from of the actual measurement from Figure 20. The relationship is exact at $n = \infty$. The graphed equations are based on the formula:

$$\int F_n(\Theta)\,d\Theta = \frac{2}{n \cdot i} \cdot \left[e^{in\left(-\frac{\theta_m}{2}+\pi\right)} - e^{in\left(\frac{\theta_m}{2}+\pi\right)} \right]$$

2. Definitions

We define the angle which creates closure for the circle to be equal to 2π, and by extension a semicircle must equal π. While this may seem like we put π into the equation and we get π out, if one examines the mathematics it is apparent that this is not the case. We simply need a symbol to represent closure of the circle in real space and that is 2π. We do not assign a value to this. It must equal the sum of the harmonic components needed to reproduce it because of the GTE.

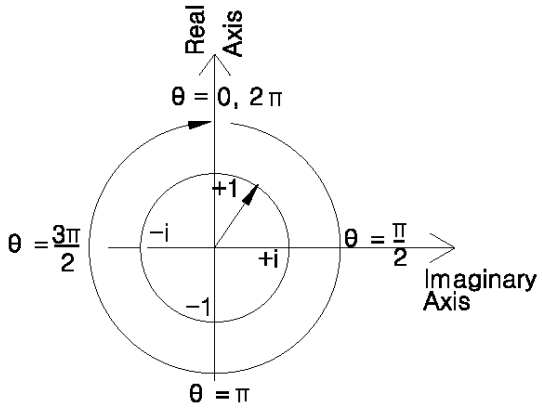


Figure 23

As part of this definition for π, we require that the circle have closure. That is to say that

$e^{i2\pi} = +1$, and by extension $e^{i\pi} = -1$, $e^{i\pi/2} = +i$, and $e^{i3\pi/2} = -i$. See Figure 23.

We stress here that we are not giving π a value. We are merely defining 2π as the angle θ which causes $e^{i\theta}$ to close. In this respect the Euler equation is simply a map laying out North, South, East and West and giving values 1, -1, $+i$ and $-i$ to define turns relative to these. The Euler equation does not give us the value of π. This is discussed in greater detail in the earlier discussion of the Euler equation.

Note that Section 4 which contains an overview of the MH equations is in many ways complete on its own. Advanced readers may wish to skip Section 3.

3. Development of Equations & Normalization

Here we develop the harmonic equations and normalization and demonstrate how the mechanics works. Let $f(\theta)$ be the function in real space for determining arc length along a circle of $r = 1$ with θ being the angle turned. Let $F(\Theta)$ be the corresponding function in harmonic space that we must determine. There is an equivalence relation between the real space and harmonic space maps of the action $A \rightarrow B$ called the General Transform Equation or GTE:

GTE
$$\int_A^B f(\theta)d\theta = \int_A^B F(\Theta)d\Theta \qquad \text{(III.A)}$$

We are looking for the function $F(\Theta)$ which solves the GTE. The integral of that function in harmonic space must have the same value as the integral of $f(\theta)$ in real space at every point on the interval. The general form of the right hand side of (III.A) is:

$$F(\Theta) = \sum_{n=0}^{\infty} a \cdot e^{in\cdot(b\Theta+c)} \qquad \text{(III.B)}$$

$F(\Theta)$ is the sum of a set of n wave functions of variable Θ and a, b, and c are normalization constants. If we imagine the circle to be a ring of solid material we can visualize the harmonic modes as vibration modes of the ring, in which the component waves must add up to create the circle in real space. See Figure 24. That is what (III.A) states mathematically.

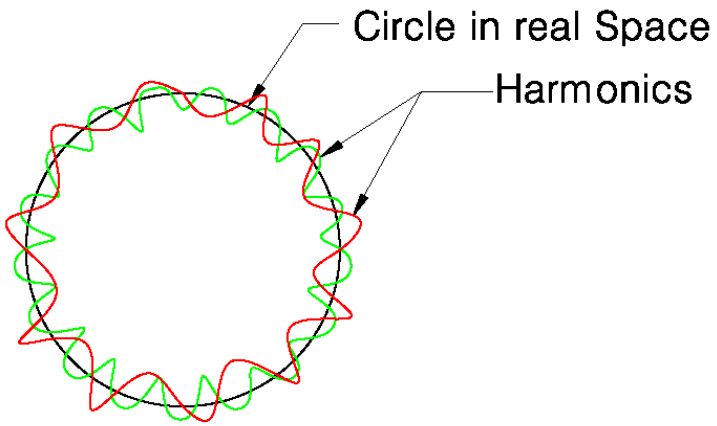


Figure 24

The harmonic modes can be thought of as sinusoidal vibrations of a ring. The harmonics sum to create a function that when integrated models π exactly.

We stress this is only an analogy. The circle does not really exhibit vibrations as the harmonics take place in virtual space.

Looking at the left side of (III.A) we see that $f(\theta)=1$ since the left side represents the action of measuring the arc length C. After all, $C=\int r\cdot d\theta$ and $r=1$.

And the GTE (III.A) can be written as

$$\int_A^B 1\cdot d\theta=\int_A^B\sum_{n=0}^{\infty}a\cdot e^{in\cdot(b\Theta+c)}d\Theta \qquad \text{(III.C)}$$

We will now make a first guess at $F(\Theta)$ ***and*** $f(\theta)$, ***then correct our assumptions based on the result.***
For our first guess we try $b=1,\quad a=2,$ and $c=0.$

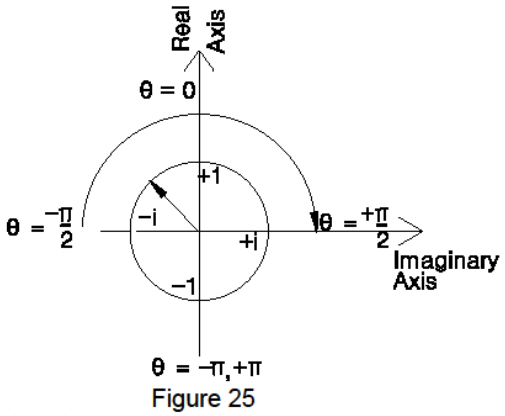


Figure 25
The boundaries of integration are -π/2 and +π/2 which is a semicircle in θ Space

Therefore our first guess is: $f(\theta)=1$ and $F(\Theta)=\sum_{n=0}^{\infty}2\cdot e^{in\cdot\Theta}$ (III.D)

Let θ_m be the specific angle measured. Then the boundary values *A* and *B* are given by $-\frac{1}{2}\theta_m$ and $+\frac{1}{2}\theta_m$ respectively. See Figure 25. We do not use 0 as a boundary of integration as this would not result in a solution to the GTE. In general harmonic analysis requires us to use negative boundaries in symmetry with positive ones.

For the left side of the GTE

$$\int_{-\theta_m/2}^{+\theta_m/2}f(\theta)d\theta=\int_{-\theta_m/2}^{+\theta_m/2}1\cdot d\theta=\theta\Big|_{-\theta_m/2}^{+\theta_m/2}=\theta_m \qquad \text{(III.E)}$$

For the right side of the GTE

$$\int_{-\theta_m/2}^{+\theta_m/2}\mathrm{F}(\Theta)d\Theta=\int_{-\theta_m/2}^{+\theta_m/2}\sum_{n=0}^{\infty}2\cdot e^{in\cdot\Theta}d\Theta=\quad\sum_{n=0}^{\infty}\frac{2}{ni}e^{in\cdot\Theta}\Big|_{-\theta_m/2}^{+\theta_m/2}=\quad\sum_{n=0}^{\infty}\frac{2}{ni}\left(e^{in\cdot\theta_m/2}-e^{-in\cdot\theta_m/2}\right) \qquad \text{(III.F)}$$

In Figure 26 we plot the left hand side of the GTE (III.F) and the first guess right hand side (III.G) on the same graph. For the latter we use the sum of the first 100 harmonics. We identify these as $LHS(\theta)$ and $RHS(\Theta)$ and they represent $\int f(\theta)d\theta$ and $\int F(\Theta)d\Theta$ respectively. Note that the left hand side represents the arc measurement of a circle of radius 1 in real space as shown in Figure 20. We are trying to find a function for the right hand side that is graphically equivalent to this.

$$LHS(\theta_m) := \theta_m \qquad RHS(\theta_m) := \sum_{n=1}^{100} \left[\frac{2}{n \cdot i} \cdot \left(e^{in\frac{\theta_m}{2}} \right) - \frac{2}{(n \cdot i)} \cdot \left(e^{-in\frac{\theta_m}{2}} \right) \right]$$

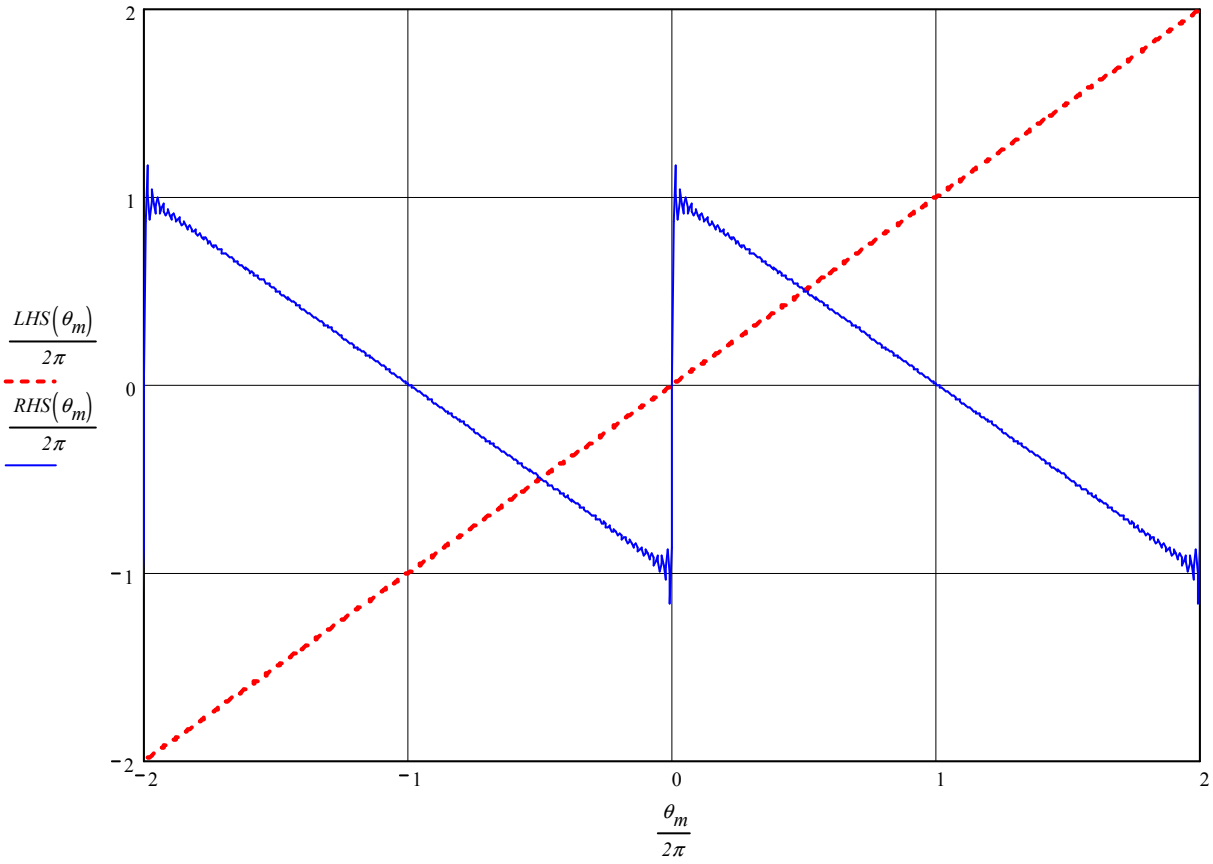


Figure 26 - The left hand side and first guess right hand side of the GTE are plotted. The left side, $\int f(\theta)d\theta$ is graphed using $f(\theta) = 1$ and appears as a straight line passing though the origin as per Figure 20. This represents arc measurement in real space. The right side, $\int F(\Theta)d\Theta$ is graphed using $F(\Theta) = \sum_{n=0}^{\infty} 2 \cdot e^{in \cdot \Theta}$ as a trial value. The first 100 harmonics were summed. It is seen that the right hand side approaches a straight line and the slope is correct but needs to be mirrored and shifted to be graphically equivalent with the left hand side.

We want the two functions to have the same value at all points in the interval. That means the two graphs need to be identical, or that $\int F(\Theta)d\Theta$ is graphically equivalent to $\int f(\theta)d\theta$.

It is seen from the figure that with only 100 harmonics $\int F(\Theta)d\Theta$ approaches a straight line. The slope is correct, but it needs to be mirrored and shifted to correctly model the real action.

Based on this result we now make a second guess. We will try $b=-1$ and $a=-2,$ and $c=\pi.$

Our second guess is: $f(\theta)=1$ and $F(\Theta)=\sum_{n=0}^{\infty}-2\cdot e^{in\cdot(-\Theta+\pi)}$ (III.G)

For the left side of the GTE

$$\int_{-\theta_m/2}^{+\theta_m/2} f(\theta)d\theta=\int_{-\theta_m/2}^{+\theta_m/2} 1\cdot d\theta=\theta\Big|_{-\theta_m/2}^{+\theta_m/2}=\theta_m \qquad \text{(III.H)}$$

For the right side of the GTE

$$\int_{-\theta_m/2}^{+\theta_m/2} \mathrm{F}(\Theta)d\Theta=\int_{-\theta_m/2}^{+\theta_m/2}\sum_{n=0}^{\infty}-2\cdot e^{in\cdot(-\Theta+\pi)}d\Theta=$$

$$\sum_{n=0}^{\infty}\frac{2}{in}e^{in\cdot(-\Theta+\pi)}\Big|_{-\theta_m/2}^{+\theta_m/2}=\sum_{n=0}^{\infty}\frac{2}{in}\left(e^{ni\cdot(\frac{-\theta_m}{2}+\pi)}-e^{ni\cdot(\frac{+\theta_m}{2}+\pi)}\right) \qquad \text{(III.i)}$$

In Figure 27 we again plot the left hand side $LHS(\theta)$, and the second trial right hand side first 100 harmonics $RHS(\Theta)$, which represent $f(\theta)$ and $F(\Theta)$ in (III.H) and (III.i).

We see from the figure that the $\int F(\Theta)d\Theta$ does converge to $\int f(\theta)d\theta$. Therefore $F(\Theta)$ is the correct function for mapping the real space function $f(\theta)$ in harmonic space. Both maps to give the same result over the entire range and are therefore graphically equivalent as $n\to\infty.$

$$LHS(\theta_m) := \theta_m \qquad RHS(\theta_m) := \sum_{n=1}^{100} \left[\frac{2}{n \cdot i} \cdot \left[e^{in \cdot \left(\frac{-\theta_m}{2} + \pi \right)} \right] - \frac{2}{n \cdot i} \cdot \left[e^{in \cdot \left(\frac{\theta_m}{2} + \pi \right)} \right] \right]$$

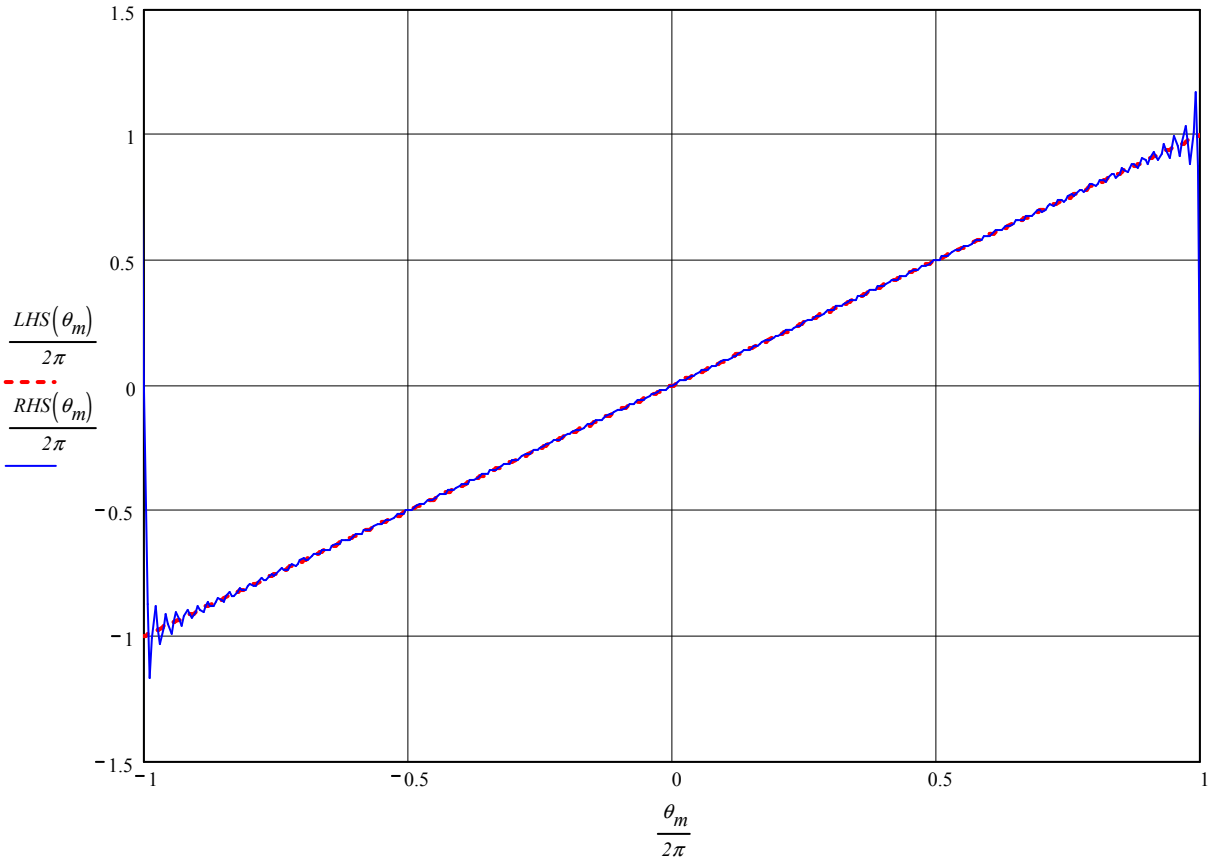


Figure 27 - $\int F(\Theta) d\Theta$ is graphed using the second trial value $F(\Theta) = \sum_{n=0}^{\infty} -2 \cdot e^{in \cdot (-\Theta + \pi)}$ summing the first 100 harmonics and appears to converge on $\int f(\theta) d\theta$ as $n \to \infty$. These functions are graphically equivalent.

In Figure 28 we demonstrate how as $n \to \infty$ the wave functions from the right side of the GTE sum to converge to the linear function (real measurement) of the left side of the GTE.

The graph has seven traces. The first trace is the left hand side of the GTE. The remaining six depict the following harmonics: 1, $1+2$, $\sum 1 \ldots 3$, $\sum 1 \ldots 10$, $\sum 1 \ldots 100$, and $\sum 1 \ldots 1,000$, the last being the thin line that is nearly straight. It closely models the left hand side. The harmonics approach the real function as $n \to \infty$.

$$IntF_1(\theta_m) := \frac{2}{1\cdot i}\cdot\left[e^{1\cdot i\cdot\left(\frac{-\theta_m}{2}+\pi\right)}\right] - \frac{2}{1\cdot i}\cdot\left[e^{1\cdot i\cdot\left(\frac{\theta_m}{2}+\pi\right)}\right]$$

$$SumF_2(\theta_m) := \sum_{n=1}^{2}\left[\frac{2}{n\cdot i}\cdot\left[e^{in\cdot\left(\frac{-\theta_m}{2}+\pi\right)}\right] - \frac{2}{n\cdot i}\cdot\left[e^{in\cdot\left(\frac{\theta_m}{2}+\pi\right)}\right]\right]$$

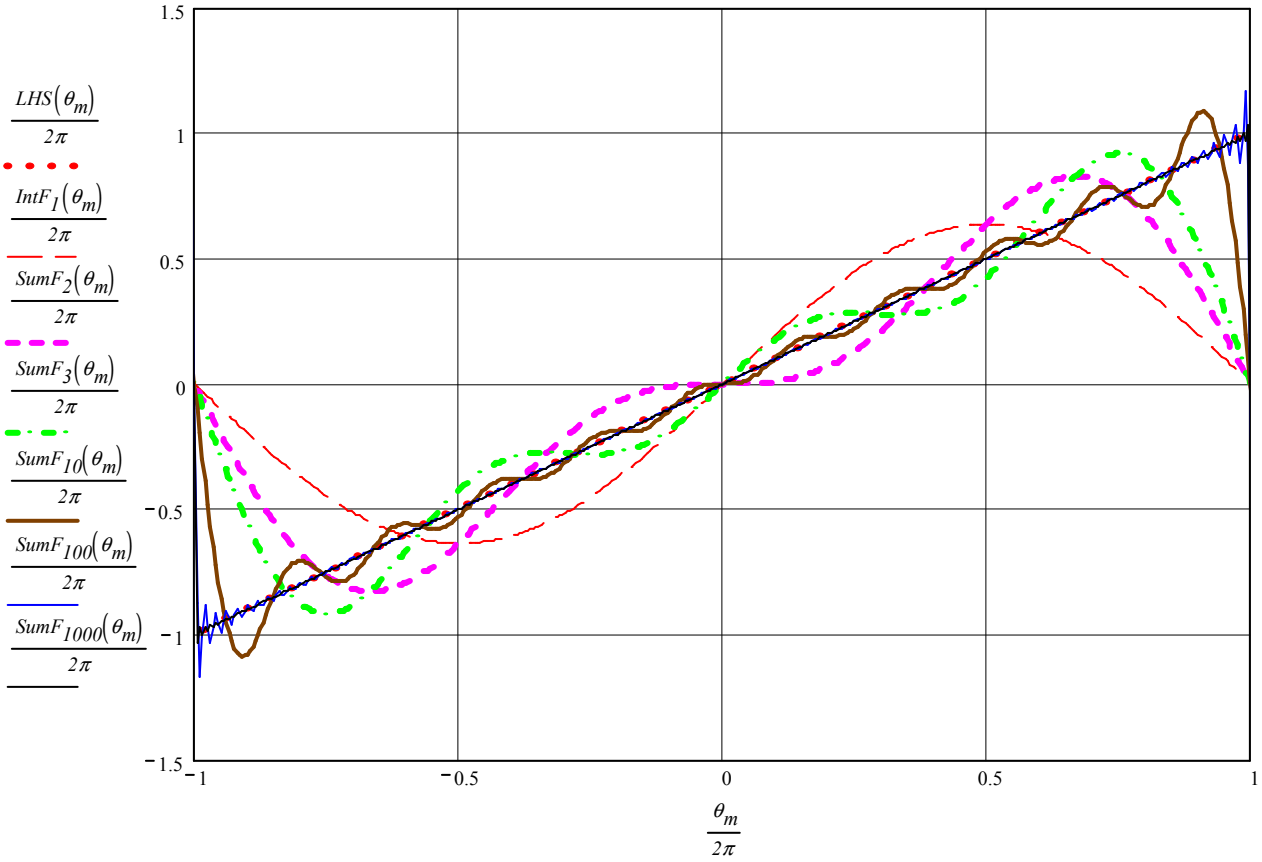


Figure 28 – Illustration of how the harmonics sum to produce an equivalent function to that found in nature. The left hand side of the GTE is shown harmonic 1 and sums of harmonics 1 to 2, 3, 10, 100, & 1,000. As more harmonics are included the sum of harmonics more closely matches the real measurement. Each subsequent harmonic decrease in amplitude as does its effect. As $n \to \infty$, $\int F(\Theta)d\Theta \to \int f(\theta)d\theta$. This indicates that $F(\Theta) = \sum_{n=0}^{\infty} -2\cdot e^{in\cdot(-\Theta+\pi)}$ satisfies the GTE.

Thus we have found the function $F(\Theta)$ that we needed to satisfy the equivalence relation.

Now we will verify that this value of $F(\Theta)$ does generate the Gregory Series when inserted into the GTE.

We again evaluate $\int f(\theta)d\theta$ at boundaries $-\pi/2$ to $+\pi/2$ to create a measured angle of $\theta_m = \pi$ as shown in Figure 25.

So for the left side of GTE :

$$\int_{-\pi/2}^{+\pi/2} f(\theta)d\theta = \int_{-\pi/2}^{+\pi/2} 1 \cdot d\theta = \theta\Big|_{-\pi/2}^{+\pi/2} = \pi \qquad \text{(III.J)}$$

For the right side, and using the identity $e^{i3\pi/2} = e^{-i\pi/2}$:

$$\int_{-\pi/2}^{+\pi/2} \mathrm{F}(\Theta)d\Theta = \int_{-\pi/2}^{+\pi/2} \sum_{n=0}^{\infty} -2 \cdot e^{in\cdot(-\Theta+\pi)}d\Theta =$$

$$\sum_{n=0}^{\infty} \frac{2}{in} e^{in\cdot(-\Theta+\pi)}\Big|_{-\pi/2}^{+\pi/2} = \sum_{n=0}^{\infty} \frac{2}{in}\left(e^{ni\cdot(\frac{-\pi}{2}+\pi)} - e^{ni\cdot(\frac{+\pi}{2}+\pi)} \right) =$$

$$\sum_{n=0}^{\infty} \frac{2}{in}\left(e^{ni\cdot(\pi/2)} - e^{ni\cdot(3\pi/2)}\right) = \sum_{n=0}^{\infty} \frac{2}{in}\left(e^{ni\cdot(\pi/2)} - e^{-ni\cdot(\pi/2)}\right) = \sum_{n=0}^{\infty} \frac{2}{in}\left(e^{ni\cdot(\pi/2)} - e^{-ni\cdot(\pi/2)}\right) \qquad \text{(III.K)}$$

Equating left (III.J) and right (III.K) sides of the GTE:

$$\pi = \sum_{n=0}^{\infty} \frac{2}{in} \cdot (e^{in\cdot(\pi/2)} - e^{-in\cdot(\pi/2)}) \qquad \text{(III.L)}$$

Since π is equal to four times the sum of the series, this implies each term of the series is given

$$f_n = \left(\frac{1}{4}\right) \cdot \left(\frac{2}{in} \cdot (e^{in\cdot(\pi/2)} - e^{-in\cdot(\pi/2)})\right) \qquad \text{(III.M)}$$

And in fact it does. Figure 29 shows how f_n generates the Gregory Series by subtracting the two functions.

We recognize the series in the right column of Figure 29 as $\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right)$. (III.N)

(To show the process the term F_n is calculated first, and then divided by 4 to determine f_n).

$$F(n) := \frac{2\cdot\left[e^{i\cdot n\cdot\left(\frac{\pi}{2}\right)}\right]}{i\cdot n} - \frac{2\cdot\left[e^{-i\cdot n\cdot\left(\frac{\pi}{2}\right)}\right]}{i\cdot n} \qquad \sum_{n=1}^{10000} F(n) = 3.141393$$

n =	A $\frac{2\cdot\left[e^{i\cdot n\cdot\left(\frac{\pi}{2}\right)}\right]}{i\cdot n}$ =	B $\frac{2\cdot\left[e^{-i\cdot n\cdot\left(\frac{\pi}{2}\right)}\right]}{i\cdot n}$ =	C F(n) =	C/4 f(n) = $\frac{F(n)}{4}$ =	*Coefficients*
1	2	-2	4	1	$= +1$
2	i	i	0	0	
3	-0.667	0.667	-1.333	-0.333	$= -\frac{1}{3}$
4	-0.5i	-0.5i	0	0	
5	0.4	-0.4	0.8	0.2	$= +\frac{1}{5}$
6	0.333i	0.333i	0	0	
7	-0.286	0.286	-0.571	-0.143	$= -\frac{1}{7}$
8	-0.25i	-0.25i	0	0	
9	0.222	-0.222	0.444	0.111	$= +\frac{1}{9}$
10	0.2i	0.2i	0	0	

Figure 29 – Calculation of (III.L) and (III.M) by tabulating each n value. Column A is the first term of Equation (III.L) . Column B is the second term. These are the two spinners we discuss later. Column C is the first minus the second as per (III.L). This is divided by four to determine the last tabulated column. We recognize these coefficients as the Gregory Series

$$\pi/4 = \left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots\right).$$

Also shown is the sum of the first 10,0000 harmonics. This approaches π as n approaches infinity.

Returning to (III.L) we now expand the summation to investigate how the mechanics operates:

$$\pi = \int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta = \sum_{n=0}^{\infty}\frac{2}{in}\cdot(e^{in\cdot(\pi/2)} - e^{-in\cdot(\pi/2)}) =$$

$$2\cdot\left[\left(\frac{1}{i}\cdot e^{1i\pi/2} - \frac{1}{i}\cdot e^{-1i\pi/2}\right) + \left(\frac{1}{2i}\cdot e^{2i\pi/2} - \frac{1}{2i}\cdot e^{-2i\pi/2}\right) + \left(\frac{1}{3i}\cdot e^{3i\pi/2} - \frac{1}{3i}\cdot e^{-3i\pi/2}\right) + \left(\frac{1}{4i}\cdot e^{4i\pi/2} - \frac{1}{4i}\cdot e^{-4i\pi/2}\right) + \ldots\right]$$

(III.O)

Recalling our definition of 2π as the angle at which the circle closes in the expression $e^{i\Theta}$, this implies that $e^{i2\pi} = +1$, and by extension $e^{i\pi} = -1$, $e^{i\pi/2} = +i$, and $e^{i3\pi/2} = -i$.

These are our "compass directions". This is simply based on our definition of 2π and does not assign a value to π. The $+1, +i, -1, -i$ pattern continues on a cycle of 2π so that $e^{i5\pi/2} = +i$, $e^{i3\pi} = -1$, $e^{i7\pi/2} = -i$, and $e^{i4\pi} = +1$.

Substituting these into the above we obtain $\pi = \int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta =$

$$2\cdot\left[\left(\frac{1}{i}\cdot(i)-\frac{1}{i}\cdot(-i)\right)+\left(\frac{1}{2i}\cdot(-1)-\frac{1}{2i}\cdot(-1)\right)+\left(\frac{1}{3i}\cdot(-i)-\frac{1}{3i}\cdot(+i)\right)+\left(\frac{1}{4i}\cdot(+1)-\frac{1}{4i}\cdot(+1)\right)+...\right] \tag{III.P}$$

which simplifies to $\pi = \int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta =$

$$2\cdot\left[\left(+1-(-1)\right)+\left(\frac{-1}{2i}-\left(\frac{-1}{2i}\right)\right)+\left(\frac{-1}{3}-\left(\frac{+1}{3}\right)\right)+\left(\frac{+1}{4i}-\left(\frac{1}{4i}\right)\right)+\left(\frac{+1}{5i}-\left(\frac{-1}{5i}\right)\right)+\left(\frac{-1}{6i}-\left(\frac{-1}{6i}\right)\right)+\left(\frac{-1}{7i}-\left(\frac{+1}{7i}\right)\right)+..\right]$$

$$=2\cdot\left[\left(+1-(-1)\right)+\left(\frac{-1}{2i}+\frac{+1}{2i}\right)+\left(\frac{-1}{3}-\frac{+1}{3}\right)+\left(\frac{+1}{4i}-\frac{+1}{4i}\right)+\left(\frac{+1}{5}-\frac{-1}{5}\right)+\left(\frac{-1}{6i}+\frac{+1}{6i}\right)+\left(\frac{-1}{7}-\frac{+1}{7}\right)+...\right] \tag{III.Q}$$

But $\left(\frac{-1}{2i}+\frac{+1}{2i}\right)=0,$ $\left(\frac{-1}{4i}+\frac{+1}{4i}\right)=0,$ $\left(\frac{-1}{6i}+\frac{+1}{6i}\right)=0,$ ....etcetera

We see that each pair of terms generated by an even value for *n* sums to zero and that when *n* is odd the components add, but also alternate in sign. Thus:

$$\pi = \int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta = 2\cdot\left[\left(+1-(-1)\right)+\left(\frac{-1}{3}-\frac{+1}{3}\right)+\left(\frac{+1}{5}-\frac{-1}{5}\right)+\left(\frac{-1}{7}-\frac{+1}{7}\right)+\left(\frac{+1}{9}-\frac{-1}{9}\right)...\right] \tag{III.R}$$

Or $$\pi = \int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta \;=\; 4\cdot\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-...\right) \tag{III.S}$$

Or $$\pi = \int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta \;=\; 4\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \tag{III.T}$$

And we have derived the Gregory Series for π as we set forth to do.

4. Modal Harmonic Equations for π

Let $f(\theta)$ be the function in real space for determining arc length along a circle of $r = 1$ with θ being the angle turned. Let $F(\Theta)$ be the corresponding function in harmonic space that we must determine. Let θ_m be the specific angle measured. Then the boundary values *A* and *B* are given by $-\frac{1}{2}\theta_m$ and $+\frac{1}{2}\theta_m$ respectively. There is an equivalence relation between real space and harmonic space maps of the action $A \rightarrow B$ called the General Transform Equation or GTE:

$$\text{GTE} \qquad \int_A^B f(\theta)d\theta = \int_A^B F(\Theta)d\Theta \qquad \text{(III.0)}$$

We are looking for the function $F(\Theta)$ which solves the GTE. The integral of that function in harmonic space must have the same value as the integral of $f(\theta)$ in real space. The general form of the right hand side is:

$$F(\Theta) = \sum_{n=0}^{\infty} a \cdot e^{in\cdot(b\cdot\Theta+c)} \qquad \text{(III.1)}$$

This is a set of harmonic wave functions where *n* is the harmonic wave number and a, b and c are normalization constants. We will use this in the right hand side of the GTE. The left hand side is the arc length measurement in real space. So $f(\theta) = 1$ since $r = 1$ and arc length equals $\int r \cdot d\theta$.

We set the limits of integration, or boundary values *A* and *B*, to $-\pi/2$ and $+\pi/2$. This creates an interval of measurement $\theta_m = \pi$ centered on zero. Then (III.0) can be written as

$$\int_{-\pi/2}^{+\pi/2} d\theta = \int_{-\pi/2}^{+\pi/2} \sum_{n=0}^{\infty} a \cdot e^{in\cdot(b\Theta+c)} d\Theta \qquad \text{(III.2)}$$

This relation defines a coding pattern for the harmonic components which allows them to combine in such a way that the two maps agree. n is the harmonic number and determines <u>*how much of each individual component is used*</u> in the sum. *b* is the frequency constant and is related to the boundary values (*A, B*). We will see that real and rational solutions exist for each *n* when

$$b \cdot \theta_m = \pm\pi \qquad \text{(III.3)}$$

Selecting these values in (III.2) results in all of the imaginary components from the right side of (III.2) to cancel out leaving only real numbers. Without the requirement of (III.3) the solutions could be complex and irrational as they could exist anywhere on the unit circle. The relation (III.3) restricts Θ to the quadrants of the circle and $e^{i\Theta}$ can only have the values $+1$ and -1. Thus solutions exist only at two compass points. This results in only integer ratios occurring, and the even numbered harmonics, which have imaginary values, all subtract out to zero. <u>*This determines which of the harmonic components are included and which are not used.*</u> These two constants together with n combine the individual wave components in the correct way to create the linear relationship of Figure 20.

a is the amplitude constant . It is a single factor that applies to the entire sum. It could double or half the value for example. So while *n* and b create the linear function, a determines the slope. The value of a is related to θ_m and b by the (III.2) with boundary values $A = -\frac{1}{2}\theta_m$ and $B = +\frac{1}{2}\theta_m$:

$$\theta_m = \frac{a}{b} \cdot (\pi / 2) \tag{III.4}$$

for equivalence in (III.2) so a must be either $+2$ or -2 in this case.

c is the phase factor. It offsets the result left or right but does not change the linear properties or slope.

We select $a = -2,\ b = -1,$ and $c = \pi$ and set the limits of integration, or boundary values *A* and *B*, to $-\pi / 2$ and $+\pi / 2.$ (See Section 6). This creates an interval of measurement $\theta_m = \pi$ centered on zero. Thus

$$f(\theta) = 1 \qquad \text{and} \qquad F(\Theta) = \sum_{n=0}^{\infty} -2 \cdot e^{in \cdot (-\Theta + \pi)} \tag{III.5}$$

Substituting into the GTE (III.0):

$$\int_{-\pi/2}^{+\pi/2} d\theta = \int_{-\pi/2}^{+\pi/2} \sum_{n=0}^{\infty} -2 \cdot e^{in \cdot (-\Theta + \pi)} d\Theta \tag{III.6}$$

So for the <u>left hand side</u> of the GTE we find:

$$\int_{-\pi/2}^{+\pi/2} f(\theta) d\theta = \int_{-\pi/2}^{+\pi/2} d\theta = \theta \Big|_{-\pi/2}^{+\pi/2} = \pi \tag{III.7}$$

Performing the integration and using $e^{+in \cdot (3\pi/2)} = e^{-in \cdot (\pi/2)}$ we find that for the <u>right hand side</u> of (III.1):

$$\int_{-\pi/2}^{+\pi/2} F(\Theta) d\Theta = \sum_{n=0}^{\infty} \frac{2}{in} \cdot e^{in \cdot (-\Theta + \pi)} \Big|_{-\pi/2}^{+\pi/2} = \sum_{n=0}^{\infty} \frac{2}{in} \cdot (e^{+in \cdot (\pi - \pi/2)} - e^{+in \cdot (\pi + \pi/2)}) =$$

$$\sum_{n=0}^{\infty} \frac{2}{in} \cdot (e^{+in \cdot (\pi/2)} - e^{+in \cdot (3\pi/2)}) = \sum_{n=0}^{\infty} \frac{2}{in} \cdot (e^{+in \cdot (\pi/2)} - e^{-in \cdot (\pi/2)}) \tag{III.8}$$

Substituting the values from (III.7) and (III.8) into (III.0):

$$\pi = \sum_{n=0}^{\infty} \frac{2}{in} \cdot \left(e^{+in\cdot(\pi/2)} - e^{-in\cdot(\pi/2)} \right) \tag{III.9}$$

In Figure 29 it is shown that this gives the correct value for π.

Now $e^{+in\cdot(\pi/2)} = \left(e^{i\cdot\pi/2}\right)^n$ and $e^{+in\cdot(3\pi/2)} = \left(e^{i\cdot\pi/2}\right)^{-n}$, so (III.9) can be rewritten as

$$\pi = 2 \cdot \sum_{n=0}^{\infty} \frac{1}{ni} \cdot \left[\left(e^{i\pi/2} \right)^{+n} - \left(e^{i\pi/2} \right)^{-n} \right] \tag{III.10A}$$

Recalling from our definitions that $e^{i\pi/2} = i$ this can be rewritten as

$$\pi = 2 \cdot \sum_{n=0}^{\infty} \frac{1}{ni} \cdot \left[(i)^{+n} - (i)^{-n} \right] \tag{III.10B}$$

But $\dfrac{(i)^n}{i} = (i)^{n-1}$ and $\dfrac{(i)^{-n}}{i} = (i)^{-n-1}$ so this can be simplified to

$$\pi = 2 \cdot \sum_{n=0}^{\infty} \frac{1}{n} \cdot \left[(i)^{n-1} - (i)^{-n-1} \right] \tag{III.10C}$$

For all even n that term cancels out. This includes $n = 0$. We will start the summation at $n = 1$. This avoids looking at the $n = 0$ term which has a $1/0$ factor. All even values cancel out including $n = 0$ so we can do this. Thus

$$\pi = 2 \cdot \sum_{n=1}^{\infty} \frac{1}{n} \cdot \left[(i)^{n-1} - (i)^{-n-1} \right] \tag{III.10D}$$

Expanding the series we obtain

$$\pi = 2 \cdot \left[1 \cdot \left((i)^0 - (i)^{-2} \right) + \frac{1}{2} \cdot \left((i)^1 - (i)^{-3} \right) + \frac{1}{3} \cdot \left((i)^2 - (i)^{-4} \right) + \frac{1}{4} \cdot \left((i)^3 - (i)^{-5} \right) + \frac{1}{5} \cdot \left((i)^4 - (i)^{-6} \right) + \ldots \right]$$

(III.10E)

Recall from Part II that $i^0 = +1$, as does i^{-8}, i^{-4}, i^{+4}, i^{+8} …. ; also that $i^1 = i$, as does i^{-7}, i^{-3}, i^{+5} …. ; $i^2 = -1$, as does i^{-6}, i^{-2}, i^{+6} ... ; and $i^3 = -i$ as does i^{-5}, i^{-1}, i^{+7} …. Therefore:

$$\pi = 2 \cdot \left[1 \cdot \left(1 - (-1) \right) + \frac{1}{2} \cdot (i - i) + \frac{1}{3} \cdot \left((-1) - (+1) \right) + \frac{1}{4} \cdot \left((-i) - (-i) \right) + \frac{1}{5} \cdot \left(1 - (-1) \right) + \ldots \right] \tag{III.10F}$$

This simplifies to

$$\pi = 2 \cdot \left[(1 \cdot 2) + \left(\frac{1}{2} \cdot 0\right) - \left(\frac{1}{3} \cdot 2\right) + \left(\frac{1}{4} \cdot 0\right) + \left(\frac{1}{5} \cdot 2\right) + \left(\frac{1}{6} \cdot 0\right) - \left(\frac{1}{7} \cdot 2\right) \ldots \right] \qquad \text{(III.10G)}$$

Or

$$\pi = 4 \cdot \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right) \qquad \text{(III.10H)}$$

This can be written as

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \qquad \text{(III.11)}$$

This is the familiar Gregory series for π.

Summary of Procedure

To better explain what we have done, first we found the harmonic space equation $F(\Theta)$ that when integrated over harmonic space $d\Theta$ produces a function that behaves identically to the real space (θ space) function $f(\theta)$ when integrated over angle $d\theta$ for the same boundary conditions $A \rightarrow B$. The harmonic space equation is, of course, composed of the myriad of possible harmonics each one at a smaller amplitude and higher frequency than the one before it. Because the amplitudes get smaller and smaller, as n approaches infinity the contribution of additional harmonics approaches zero. Thus the infinite sum of waves $\int F(\Theta) d\Theta$ converges to the physical measurement $\int f(\theta) d\theta$. In practice we need only calculate only to the accuracy required.

When we establish the limits of integration $-\pi / 2$ and $+\pi / 2$ we are setting the boundary conditions of a differential equation that is written in integral form. This equation has eigenvalues which determine the pattern that needs to be applied to the harmonics to solve the differential equation. Like a vibrating string fixed at each end, the boundary conditions set rules on the permissible modes. In our case all of the terms where n is even cancel out, while the odd terms have a factor of $2 / n$ and alternate in sign.

Although we cannot view the harmonic space, the setting of boundary conditions establishes eigenvalues for the differential equation. It is a requirement of physics that mapping in the two systems yield the same result. The sum of the eigenvalues from harmonic space appears as a remnant in real space as π. The great paradox is that we cannot view harmonic space yet it determines the value of our fundamental constants.

In summary the arc length around a circle with $r=1$ is given by $\int_{-\pi/2}^{+\pi/2} f(\theta)d\theta = \pi$ and must equal $\int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta$ for the two mapping spaces to have the same result for a common event, and since

$$\int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta = -4\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \tag{III.12}$$

$$\text{then } \pi \quad \underline{\text{must equal}} \quad 4\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \tag{III.13}$$

Summary of Normalized Equations

With the normalization factors $a=-2,\ b=-1$, and $c=\pi$ determined, and $\theta_m=\pi$, $A=-\pi/2$ and $B=+\pi/2$ now established, we restate:

Returning to the GTE $$\int_A^B f(\theta)d\theta = \int_A^B F(\Theta)d\Theta \tag{III.14}$$

The left and right sides are of the form

$$f(\theta)=1 \qquad \text{and} \qquad F(\Theta)=\sum_{n=0}^{\infty} a\cdot e^{in\cdot(b\Theta+c)} \tag{III.14A}$$

Substituting $$\int_A^B f(\theta)d\theta = \int_A^B \sum_{n=0}^{\infty} a\cdot e^{in\cdot(b\Theta+c)}d\Theta \tag{III.15}$$

And with $$f(\theta)=1 \qquad \text{and} \qquad F(\Theta)=\sum_{n=0}^{\infty} -2\cdot e^{in\cdot(-\Theta+\pi)} \tag{III.16}$$

Substitution yields $$\int_{-\pi/2}^{+\pi/2} d\theta = \int_{-\pi/2}^{+\pi/2}\sum_{n=0}^{\infty} -2\cdot e^{in\cdot(-\Theta+\pi)}d\Theta \tag{III.17}$$

For the left hand side of the GTE:

$$\int_{-\pi/2}^{+\pi/2} f(\theta)d\theta = \int_{-\pi/2}^{+\pi/2} 1\cdot d\theta = \theta\Big|_{-\pi}^{+\pi} = \pi \tag{III.18}$$

For the right hand side we obtain:

$$\int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta = \sum_{n=0}^{\infty} \frac{2}{in} \cdot e^{in\cdot(-\Theta+\pi)}\Big|_{-\pi}^{+\pi} = \sum_{n=0}^{\infty} \frac{2}{in} \cdot (e^{+in\cdot(\pi/2)} - e^{-in\cdot(\pi/2)}) \qquad \text{(III.19)}$$

The expansion of this was shown in Equations (III.O) through (III.R) and is condensed here:

$$\int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta = \sum_{n=0}^{\infty} \frac{2}{in} \cdot (e^{in\cdot(\pi/2)} - e^{-in\cdot(\pi/2)}) =$$

$$2 \cdot \left[\left(\frac{1}{1i} \cdot e^{1i\pi/2} - \frac{1}{1i} \cdot e^{-1i\pi/2} \right) + \left(\frac{1}{2i} \cdot e^{2i\pi/2} - \frac{1}{2i} \cdot e^{-2i\pi/2} \right) + \left(\frac{1}{3i} \cdot e^{3i\pi/2} - \frac{1}{3i} \cdot e^{-3i\pi/2} \right) + \left(\frac{1}{4i} \cdot e^{4i\pi/2} - \frac{1}{4i} \cdot e^{-4i\pi/2} \right) + .. \right]$$

$$= 2 \cdot \left[\left(+1 - (-1) \right) + \left(\frac{-1}{3} - \frac{+1}{3} \right) + \left(\frac{+1}{5} - \frac{-1}{5} \right) + \left(\frac{-1}{7} - \frac{+1}{7} \right) + \left(\frac{+1}{9} - \frac{-1}{9} \right) ... \right]$$

$$= 4 \cdot \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - ... \right) = 4 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \qquad \text{(III.20)}$$

So for the left side of the GTE $$\int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta = 4 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \qquad \text{(III.21)}$$

Substituting left (III.18) and right (III.21) sides into the GTE (III.14):

$$\pi = 4 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \qquad \text{(III.22)}$$

This can be rearranged to $$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \qquad \text{(III.23)}$$

and we have the Gregory Series for $\pi / 4$.

5. Mechanics

How does the math work? Is there a mechanism that models all of this? The upper and lower boundaries $+\pi/2$ and $-\pi/2$ of the integral along with the constant $b=-1$ result in the factor $(e^{i\cdot\pi/2})^n-(e^{i\cdot\pi/2})^{-n}$ in Equation (III.10A). These two terms which are subtracted can be thought of as two separate spinners that are synchronized to move a quarter turn with each n, but they are moving in opposite directions. See Figure 30.

At the first harmonic $n=1$ the two spinners point towards each other, one with a value of $+i$ and the other $-i$. So $(+i)-(-i)=2i$. The factor $(e^{i\cdot\pi/2})^n-(e^{i\cdot\pi/2})^{-n}$ requires that we subtract the results from the two spinners.

For $n=2$ they point both point down, each with a value of -1 So $(-1)-(-1)=0$.
For $n=3$ they point away from each other, one with a value of $-i$ and the other $+i$. So $(-i)-(+i)=-2i$.
For $n=4$ they point both point up, each with a value of +1 So $(+1)-(+1)=0$.
At $n=5$ the pattern repeats to that of $n=1$.

We are subtracting the result of one spinner from the other. Therefore when both spinners point in the same direction those terms will cancel. This occurs for all of the even numbered harmonics as both spinners are in the same position, either up or down.

For the odd numbered harmonics the values of the two spinners are always of opposite sign. These are subtracted and therefore result in a non-zero value. At $n=1$ we subtract a negative value from a positive value and the result is positive. But at $n=3$ we subtract a positive value from a negative value and the result is negative. Therefore the odd numbered harmonics will alternate in sign.

The spinners track and determine which modes add, which subtract, and which cancel out. All the even numbered harmonics cancel out and the odd numbers double and alternate in sign. The amplitude decreases for each harmonic by a factor of $1/n$.

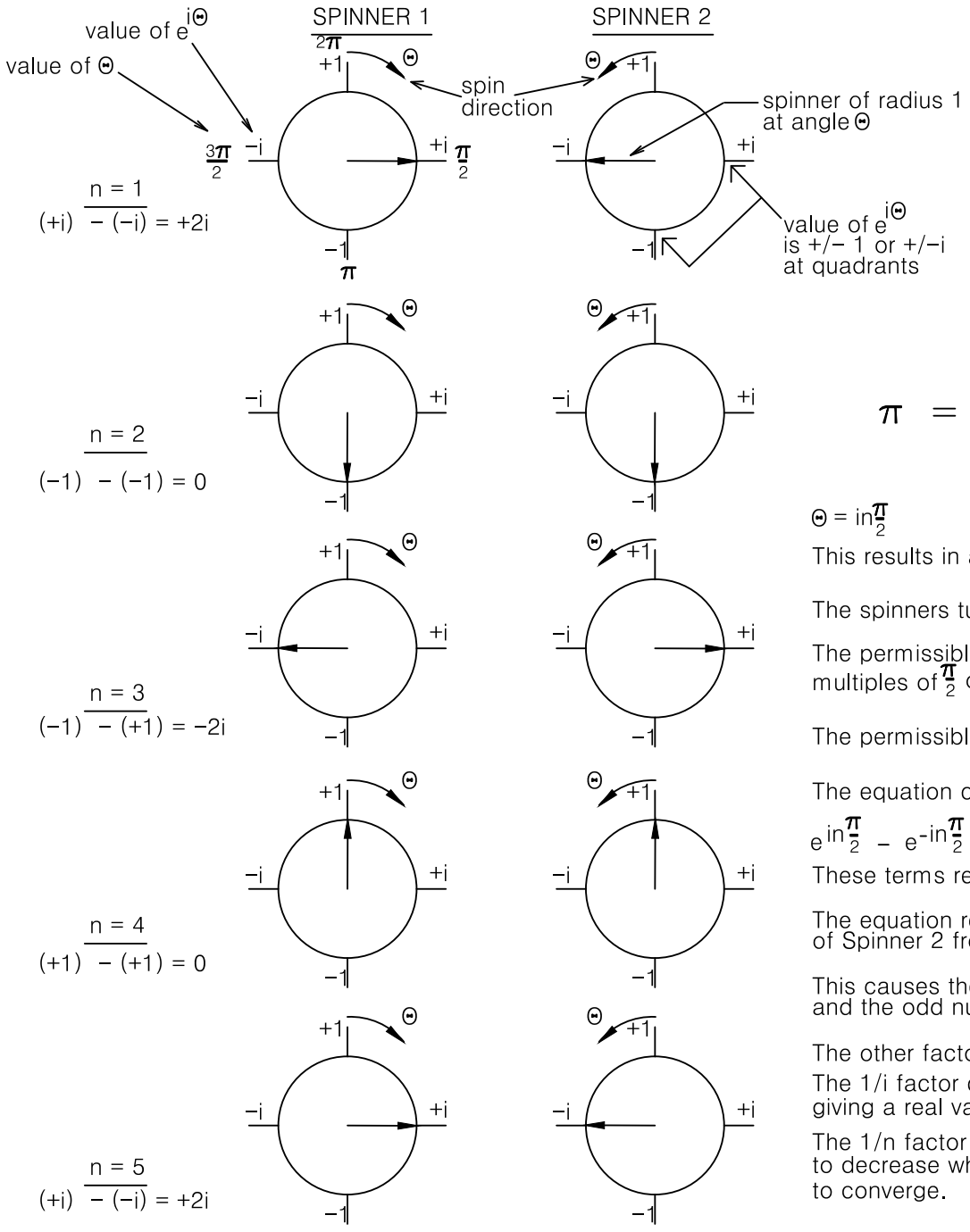
We are solving a differential equation that appears in integral form. It defines which harmonics to include and how much of each one in order to have the results in harmonic space agree with those of nature. There is a complete range of harmonics and all amplitudes are permissible, yet there is only a single solution.

The exact recipe for the amount of each mode that needs to be combined is encoded in the differential equation. It is a code that when applied to the harmonic spectrum results in the value of π. This is nature's means for storing that value and accessing it.

π has to have some value and there are insufficient dimensions in real space to provide this. Nature chose this method to establish that value. Whenever an action occurs involving the value of π that action is following natural law and obeying a differential equation in the same way that a falling body would follow its natural laws.

But π it is more subtle than G. π is dimensionless so there are no indications of what to associate it to. Our assumption is that it is the set of waves that is needed to create it. We have a model that is relatively easy to visualize with two opposing spinners that operate in virtual space.

CODING PATTERN FROM TWO OPPOSING SPINNERS



SPINNER 1 SPINNER 2

$$\pi = \sum_{n=1}^{\infty} \frac{2}{in}\left(\left(e^{i\frac{\pi}{2}}\right)^{n} - \left(e^{i\frac{\pi}{2}}\right)^{-n}\right)$$

$\Theta = in\frac{\pi}{2}$

This results in a quarter turn for each n

The spinners turn in opposite directions.

The permissible positions for the spinner are multiples of $\frac{\pi}{2}$ or vertical and horizontal only.

The permissible values for $e^{i\Theta}$ are +1, +i, −1, −1.

The equation contains the factor

$e^{in\frac{\pi}{2}} - e^{-in\frac{\pi}{2}}$ or equally $\left(e^{i\frac{\pi}{2}}\right)^{n} - \left(e^{i\frac{\pi}{2}}\right)^{-n}$

These terms represent the two opposing spinners.

The equation requires we subtract the result of Spinner 2 from Spinner 1.

This causes the even numbers to cancel out and the odd numbers to add but alternate in sign.

The other factor in the equation is 1/ni.

The 1/i factor cancels with i from the odd harmonics giving a real value.

The 1/n factor causes the contribution of each mode to decrease which is neccessary for the infinite sum to converge.

FIGURE 30

6. More on Normalization

The combination of normalization factors a, b, and c and measured angle θ_m which were used are not the only ones possible for generating this series. Other combinations will also generate the series.

θ_m determines the boundary values of the integral, $A = -\theta_m / 2$ and $B = +\theta_m / 2$. When $\theta_m = \pi$ and $b = -1$ this results in $\int_A^B \sum_{n=0}^{\infty} a \cdot e^{in\cdot(\mathrm{b}\Theta + c)} d\Theta$ producing the factor $e^{in\cdot(\pi/2)} - e^{-in\cdot(\pi/2)}$ which generates the series. This restricts the evaluation to spinner positions up and down only. Here the imaginary terms cancel out leaving only real numbers, and the ± 1 spinner value further restricts the evaluation to rational numbers only.

It was previously indicated that the normalization factors for this series follow the relation $b \cdot \theta_m = \pm\pi$. Other combinations of θ_m and b will also generate the series or a multiple of it. The amplitude and phase factors would be adjusted accordingly to result in graphical equivalence.

7. Generation of Other Series

Is it possible to use other patterns where the spinners do not land at the quadrants?

If this were to occur we would get irrational values for $e^{in\cdot(\Theta)}$ rather than $+1, -1, +i, -i$ only. We will try other values for b. In Figure 31 we compare $b = 1/2$ to $b = 1/3$ and $b = 1/4$.

We see that for $b = 1/2$: $\sum\int F_n = \pi$; For $b = 1/3$: $\sum\int F_n = 2\pi$; and for $b = 1/4$: $\sum\int F_n = 3\pi$. The value is one less than the reciprocal, times π. In general, $\sum\int F_n = (\mathrm{b}^{-1} - 1)\pi$.

Also we see that using a different value for b can generate a different series for π. For example $b = 1/3$ will generate the series:

$$\frac{2\pi}{3\sqrt{3}} = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - .... = \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{3n+1} + \frac{(-1)^n}{3n+2}\right] \qquad \text{(III.Z)}$$

It is interesting to note that in this series every third term equals zero. There are two terms of one sign, then zero, then two terms of the opposite sign. This occurs because the spinners land at intervals with value $\pm\sqrt{3}/2$, so the first two are on the right half of the circle (+ value), the third is down (0), and the next two are on the left side (- value). See Figure 32.

b=1/2

1. The integral on the right hand side of the GTE is:

$$IntF_n := \frac{2\left\lfloor e^{i\cdot n\cdot\left(\frac{\pi}{2}\right)}\right\rfloor}{i\cdot n} - \frac{2\left\lfloor e^{-i\cdot n\cdot\left(\frac{\pi}{2}\right)}\right\rfloor}{i\cdot n}$$

2. The value of this integral is:

$$\sum_{n=1}^{1000}\left[\frac{2\left\lfloor e^{i\cdot n\cdot\left(\frac{\pi}{2}\right)}\right\rfloor}{i\cdot n} - \frac{2\left\lfloor e^{-i\cdot n\cdot\left(\frac{\pi}{2}\right)}\right\rfloor}{i\cdot n}\right] = 3.14 \qquad = 1\pi$$

3. The equation also provides the combination of harmonic components are for each *n*:

$$\frac{IntF_n}{4} =$$

1
0
-0.333
0
0.2
0
-0.143
0
0.111
0

4. This combination can be written as a series:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \ldots.$$

5. This can be expressed in summation form:

$$\frac{\pi}{4} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

b=1/3

$$InfF_n := \frac{3\left\lfloor e^{i\cdot n\cdot\left(\frac{\pi}{3}\right)}\right\rfloor}{i\cdot n} - \frac{3\left\lfloor e^{-i\cdot n\cdot\left(\frac{\pi}{3}\right)}\right\rfloor}{i\cdot n}$$

$$\sum_{n=1}^{1000}\left[\frac{3\left\lfloor e^{i\cdot n\cdot\left(\frac{\pi}{3}\right)}\right\rfloor}{i\cdot n} - \frac{3\left\lfloor e^{-i\cdot n\cdot\left(\frac{\pi}{3}\right)}\right\rfloor}{i\cdot n}\right] = 6.283 \qquad = 2\pi$$

$$\frac{InfF_n}{3\sqrt{3}} =$$

1
0.5
0
-0.25
-0.2
0
0.143
0.125
0
-0.1

$$\frac{2\pi}{3\sqrt{3}} = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \ldots.$$

$$\frac{2\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{3n+1} + \frac{(-1)^n}{3n+2}\right]$$

b=1/4

$$IntF_n := \frac{4\left\lfloor e^{i\cdot n\cdot\left(\frac{\pi}{4}\right)}\right\rfloor}{i\cdot n} - \frac{4\left\lfloor e^{-i\cdot n\cdot\left(\frac{\pi}{4}\right)}\right\rfloor}{i\cdot n}$$

$$\sum_{n=1}^{1000}\left[\frac{4\left\lfloor e^{i\cdot n\cdot\left(\frac{\pi}{4}\right)}\right\rfloor}{i\cdot n} - \frac{4\left\lfloor e^{-i\cdot n\cdot\left(\frac{\pi}{4}\right)}\right\rfloor}{i\cdot n}\right] = 9.415 \qquad = 3\pi$$

$$\frac{IntF_n}{4\sqrt{2}} =$$

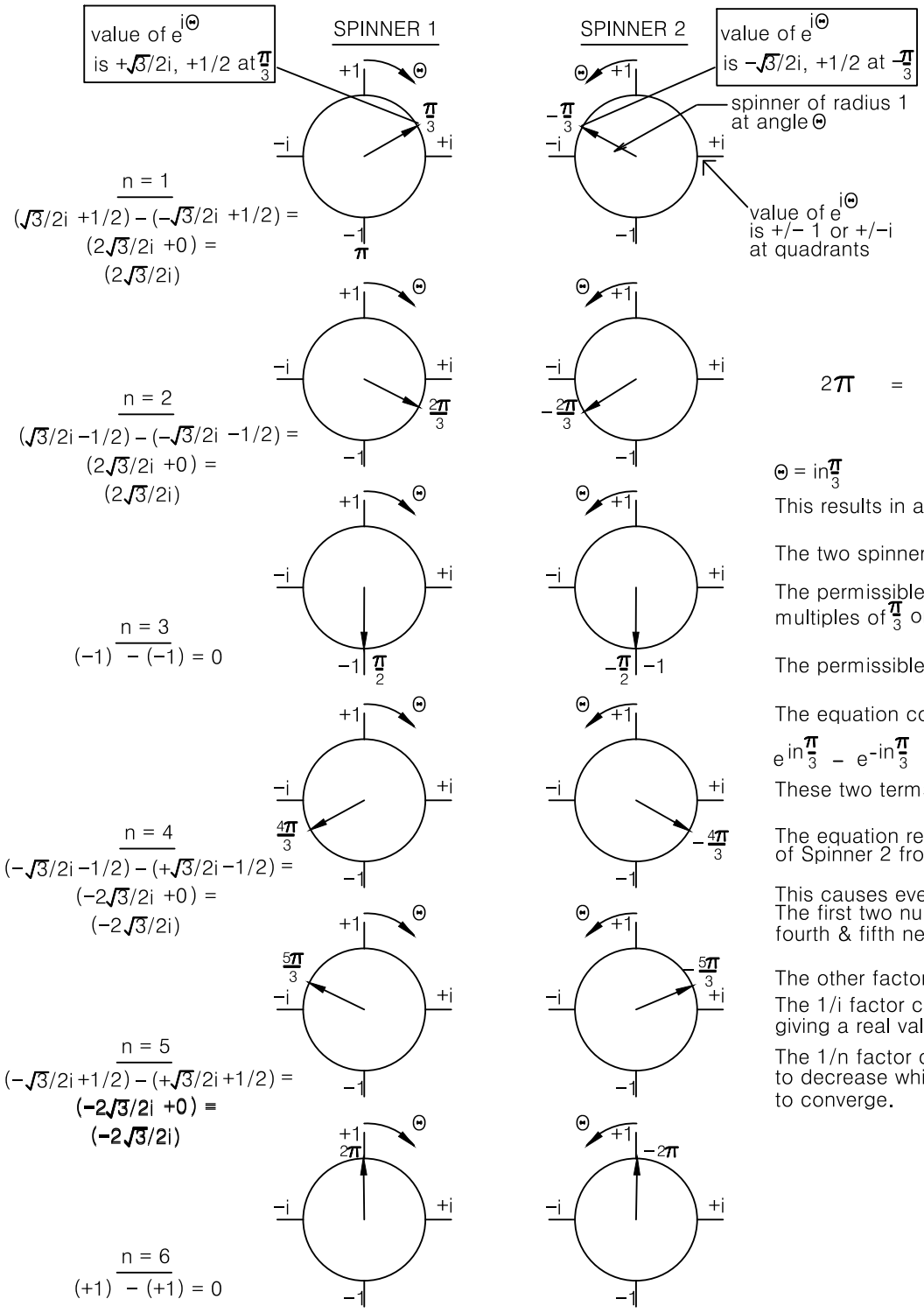
1
0.707
0.333
0
-0.2
-0.236
-0.143
0
0.111
0.141

$$\frac{3\pi}{4\sqrt{2}} = 1 + \frac{\sqrt{2}}{2} + \frac{1}{3} - \frac{1}{5} - \frac{\sqrt{2}}{6} - \frac{1}{7} + \ldots.$$

$$\frac{3\pi}{4\sqrt{2}} = \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{4n+1} + \frac{\sqrt{2}\cdot(-1)^n}{4n+2} + \frac{(-1)^n}{4n+3}\right]$$

Figure 31 - $b = 1/2,\ 1/3$ and $1/4$ in equation (III.W) result in different series for π.

CODING PATTERN FROM TWO OPPOSING SPINNERS



SPINNER 1 SPINNER 2

$$2\pi = \sum_{n=1}^{\infty} \frac{3}{in}\left(\left(e^{i\frac{\pi}{3}}\right)^{n} - \left(e^{i\frac{\pi}{3}}\right)^{-n}\right)$$

$\Theta = in\frac{\pi}{3}$

This results in a one sixth turn for each n.

The two spinners turn in opposite directions.

The permissible positions for the spinner are multiples of $\frac{\pi}{3}$ or 2'oclock, 4, 6, 8, 10, & 12 only.

The permissible values for $e^{i\Theta}$ are +1, +i, −1, −1.

The equation contains the factor

$e^{in\frac{\pi}{3}} - e^{-in\frac{\pi}{3}}$ or equally $\left(e^{i\frac{\pi}{3}}\right)^{n} - \left(e^{i\frac{\pi}{3}}\right)^{-n}$

These two terms represent the two spinners.

The equation requires we subtract the result of Spinner 2 from Spinner 1.

This causes every third number to cancel out. The first two numbers are positive, third zero, fourth & fifth negative, sixth zero...

The other factor in the equation is 1/ni.

The 1/i factor cancels with i from the non zero harmonics giving a real value.

The 1/n factor causes the contribution of each mode to decrease which is neccessary for the infinite sum to converge.

FIGURE 32

Part IV. Generation of Some π Squared Series

Here we generate other well known series for π squared:

$$\frac{\pi^2}{8}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\ ,\quad \frac{\pi^2}{12}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\ ,\quad \text{and}\quad \frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2} \tag{IV.A}$$

Using the same procedure, we seek the function $F(\Theta)$ that solves the GTE for π^2.

$$\int_A^B\int f(\theta)\cdot d\theta\cdot d\theta=\int_A^B\int \mathrm{F}(\Theta)\cdot d\Theta\cdot d\Theta \tag{IV.B}$$

In the previous analysis we used $f(\theta)=1$ with $F(\Theta)=\sum_{n=0}^{\infty}-2\cdot e^{in\cdot(-\Theta+\pi)}$ and $\theta_m=\pi$.

Here we will use $f(\theta)=1$ with $F(\Theta)=\sum_{n=0}^{\infty}4\cdot e^{in\cdot(-\Theta+\pi)}\cdot\left(i^n\right)$ and $\theta_m=\pi$. (IV.C)

Figure 33 demonstrates graphical equivalence and Figure 34 shows how the terms work. For the left hand side of the GTE:

$$\int_A^B\int f(\theta)d\theta=\int_{-\theta_m/2}^{\theta_m/2}\int d\theta\cdot d\theta=\theta^2\Big|_{-\theta_m/2}^{+\theta_m/2}=\theta_m{}^2 \tag{IV.D}$$

For the right hand side of the GTE: $$\int_{-\theta_m/2}^{\theta_m/2}\int\sum_{n=0}^{\infty}4\cdot e^{in\cdot(-\Theta+\pi)}\cdot\left(i^n\right)d\Theta^2 \tag{IV.E}$$

$$\sum_{n=0}^{\infty}\frac{-4\cdot e^{in\cdot(-\Theta+\pi)}}{n^2}\cdot\left(i^n\right)\Big|_{-\theta_m/2}^{+\theta_m/2}=\quad -4\cdot\sum_{n=1}^{\infty}\frac{e^{in\cdot(-\theta_m/2+\pi)}-e^{in\cdot(+\theta_m/2+\pi)}}{n^2}\cdot\left(i^n\right)$$

The even terms will cancel and we can begin summation at $n=1$. Substituting (IV.D)&(IV.E) into (IV.B) and $\theta_m=\pi$:

$$\pi^2=-4\cdot\sum_{n=1}^{\infty}\frac{e^{in\cdot(-\pi/2+\pi)}-e^{in\cdot(+\pi/2+\pi)}}{n^2}\cdot\left(i^n\right) \tag{IV.F}$$

Rearranging $$\frac{\pi^2}{4}=-\sum_{n=1}^{\infty}\frac{e^{in\cdot(\pi-\pi/2)}-e^{in\cdot(\pi+\pi/2)}}{n^2}\cdot\left(i^n\right) \tag{IV.G}$$

Expanding the series and eliminating the zero terms:

$$\frac{\pi^2}{4} = -\left[\left((+i)-(-i)\right)\cdot(i) + \left(\frac{(-1)-(-1)}{4}\right)\cdot(-1) + \left(\frac{(-\mathrm{i})-(+\mathrm{i})}{9}\right)\cdot(-i) + \left(\frac{(+1)-(+1)}{16}\right)\cdot(-i) + ... \right]$$

$$= +2\left[1+0+\frac{1}{9}+0+\frac{1}{25}+0+\frac{1}{49}+.... \right] \qquad \text{(IV.H)}$$

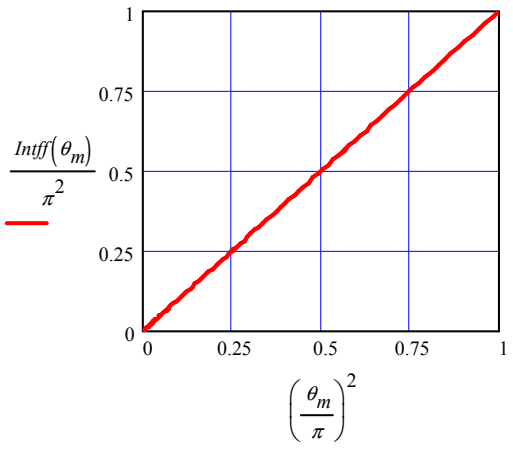
And we find $$\frac{\pi^2}{8} = 1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+.... = \sum_{n=0}^{\infty}\frac{1}{(2n-1)^2}. \qquad \text{(IV.i)}$$

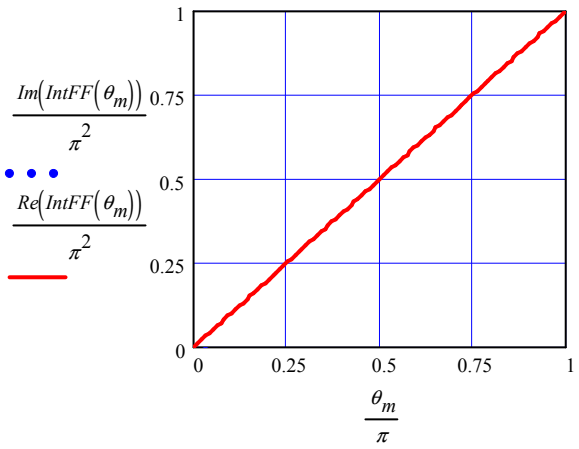
This is the first of the relations we set out to derive. We see that all of the even terms drop out and the odd terms have the factor $\frac{1}{(2n-1)^2}$.

Figure 33 – Graphical Equivalence for the π Squared Function (IV.i). Graphs of the left and right hand side of the GTE are shown. θ_m is the measured angle. When $\theta_m = \pi$, $A = -\pi/2$ and $B = +\pi/2$.

$$Intff(\theta_m) := \theta_m{}^2$$

$$IntFF(\theta_m) := \sum_{n=1}^{1000}\left[\left[\frac{-4\left\lfloor e^{\left\lfloor \mathrm{i}\cdot n\cdot\left(\pi-\frac{\theta_m}{2}\right)\right\rfloor}\right\rfloor}{n^2} - \frac{-4\left\lfloor e^{\left\lfloor \mathrm{i}\cdot n\cdot\left(\pi+\frac{\theta_m}{2}\right)\right\rfloor}\right\rfloor}{n^2}\right]\cdot\left(\mathrm{i}^{n}\right)\right]$$





The right side is plotted against $\frac{\Theta}{\pi}$ rather than $\frac{\Theta^2}{\pi^2}$ because Θ appears in the equation as a first order variable.

Figure 34 – Calculations for the π squared Equation

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n-1)^2}$$

1. The integral on the right hand side of the GTE is:

$$IntFF_n := \left[\frac{-4\left[e^{i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)}\right]}{n^2} - \frac{-4\left[e^{i\cdot n\cdot\left(\pi+\frac{\pi}{2}\right)}\right]}{n^2} \right] \cdot \left(i^{n}\right)$$

2. The value of this integral is:

$$\sum_{n=1}^{1000} \left[\left[\frac{-4\left[e^{i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)}\right]}{n^2} - \frac{-4\left[e^{i\cdot n\cdot\left(\pi+\frac{\pi}{2}\right)}\right]}{n^2} \right] \cdot \left(i^{n}\right) \right] = 9.866 = \pi^2$$

3. The equation also provides the combination of harmonic components for each *n*:

$\frac{IntFF_n}{8} =$

1
0
0.111
0
0.04
0
0.02
0
0.012

4. This combination can be written as a series:

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...$$

5. This can be expressed in summation form:

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n-1)^2}$$

6. The integral on the left hand side of the GTE = -1/2 π^2 was factored out.

Note how the term $\left(i^n\right)$ works to adjust the sign of the non-zero factors:

A - B = C x D = E

$\frac{\frac{-4\left[e^{i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)}\right]}{n^2}}{8} =$

-0.5i
0.125
0.056i
-0.031
-0.02i
0.014
0.01i
$-7.813\cdot10^{-3}$
$-6.173i\cdot10^{-3}$

$\frac{\frac{-4\left[e^{i\cdot n\cdot\left(\pi+\frac{\pi}{2}\right)}\right]}{n^2}}{8} =$

0.5i
0.125
$-0.056i$
-0.031
0.02i
0.014
$-0.01i$
$-7.813\cdot10^{-3}$
$6.173i\cdot10^{-3}$

$$\frac{\left[\frac{-4\left[e^{i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)}\right]}{n^2} - \frac{-4\left[e^{i\cdot n\cdot\left(\pi+\frac{\pi}{2}\right)}\right]}{n^2}\right]}{8} =$$

-i
0
0.111i
0
-0.04i
0
0.02i
0
-0.012i

$i^n =$

i
-1
-i
1
i
-1
-i
1
i

$\frac{IntFF_n}{8} =$

1
0
0.111
0
0.04
0
0.02
0
0.012

For the second series we replace n^2 with $(n+1)^2$ in the denominator of (IV.F) and normalize. Note the sign changing term $\left(i^n\right)$ is not needed here:

$$\pi^2 = -24i \cdot \sum_{n=1}^{\infty} \frac{e^{in\cdot(-\pi/2+\pi)} - e^{in\cdot(+\pi/2+\pi)}}{(n+1)^2} \qquad \text{(IV.J)}$$

Expanding the series and eliminating the zero terms:

$$\frac{\pi^2}{24} = -i \cdot \left[\left(\frac{(+i)-(-i)}{2^2}\right)+\left(\frac{(-1)-(+1)}{3^2}\right)+\left(\frac{(-i)-(+i)}{4^2}\right)+\left(\frac{(+1)-(-1)}{5^2}\right)....\right]$$

$$= -i \cdot \left[\left(\frac{2i}{4}\right)+0+\left(\frac{-2i}{16}\right)+0+\left(\frac{2i}{36}\right)....\right] \qquad = 2\left[1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}....\right] \qquad \text{(IV.K)}$$

And we find our second series $\frac{\pi^2}{12} = 1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+.... = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n^2}\cdot$ (IV.L)

In Figure 35 we demonstrate graphical equivalence and Figure 36 contains the calculation.

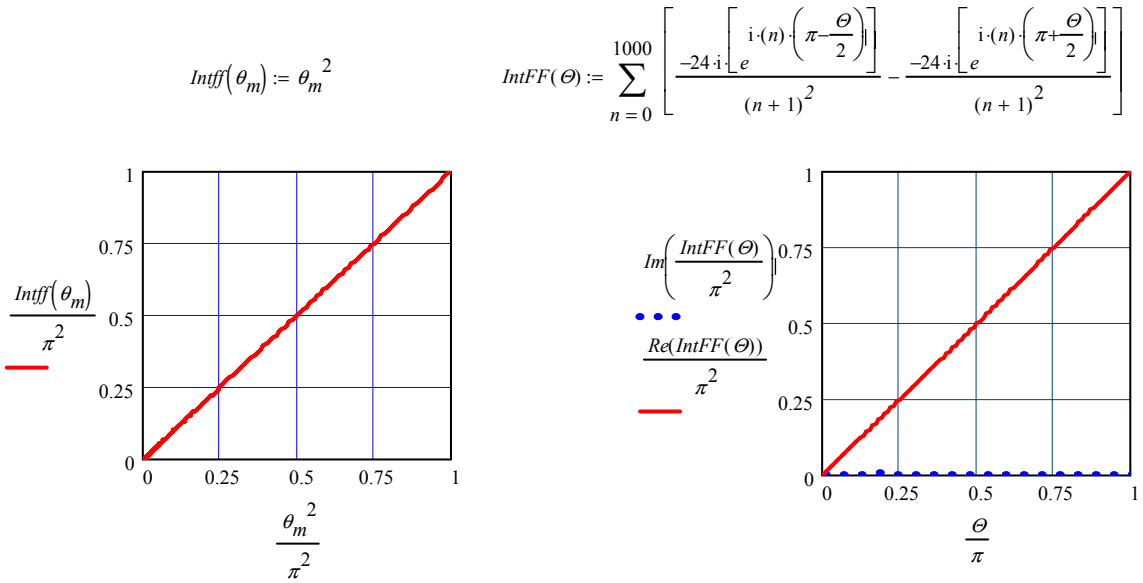


Figure 35 – Graphical Equivalence for (IV.J)

Figure 36 – Calculations for the π^2 Equation (IV.J)

1. The integral on the right hand side of the GTE is:

$$IntFF_n := \frac{-24\cdot i\cdot\left[e^{i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)}\right]}{(n+1)^2} - \frac{-24\cdot i\cdot\left[e^{i\cdot n\cdot\left(\pi+\frac{\pi}{2}\right)}\right]}{(n+1)^2}$$

2. The value of this integral is:

$$\sum_{n=1}^{1000}\left[\frac{-24\cdot i\cdot\left[e^{i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)}\right]}{(n+1)^2} - \frac{-24\cdot i\cdot\left[e^{i\cdot n\cdot\left(\pi+\frac{\pi}{2}\right)}\right]}{(n+1)^2}\right] = 9.87 \qquad = \pi^2$$

3. The equation also provides the combination of harmonic components for each *n*:

$\frac{-24\cdot i\cdot\left[e^{i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)}\right]}{\frac{(n+1)^2}{12}}$	$\frac{-24\cdot i\cdot\left[e^{i\cdot n\cdot\left(\pi+\frac{\pi}{2}\right)}\right]}{\frac{(n+1)^2}{12}}$	$\frac{IntFF_n}{12} =$
0.5	-0.5	1
0.222i	0.222i	0
-0.125	0.125	-0.25
-0.08i	-0.08i	0
0.056	-0.056	0.111
0.041i	0.041i	0
-0.031	0.031	-0.063
-0.025i	-0.025i	0
0.02	-0.02	0.04
0.017i	0.017i	0

4. This combination can be written as a series:

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots.$$

5. This can be expressed in summation form:

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

6. The integral on the left hand side of the GTE = π^2 was factored out.

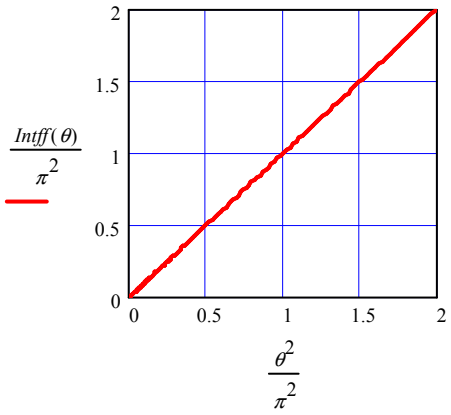
The third series is the famous Basel Problem. We note that we can obtain this series by sign changing every $4n-3$ term in the final table in Figure 36. We can do this by introducing the sign changing term $\left(\frac{i^{n-1}-i^{-n-1}}{2}\right)$. This is more complex looking than a $\left(i^{n-1}\right)$ formula but it zeros at the even n. Substituting into (IV.J) and renormalizing:

$$\pi^2 = -12i\cdot\sum_{n=1}^{\infty}\frac{e^{in\cdot(-\pi/2+\pi)} - e^{in\cdot(+\pi/2+\pi)}}{(n+1)^2}\cdot\left(\frac{i^{n-1}-i^{-n-1}}{2}\right) \qquad \text{(IV.M)}$$

$\left(\frac{i^{n-1}-i^{-n-1}}{2}\right) =$
1
0
-1
0
1
0
-1
0
1
0

In Figure 37 (IV.M) is graphed and compared to the real function. As can be seen (IV.M) ***is not*** graphically equivalent over the range. However it is graphically equivalent at $\Theta = \pi$. We can use this provided we restrict ourselves to this value only.

$$Intff(\theta) := \theta^2 \qquad IntFF(\Theta) := \sum_{n=0}^{1000} \left[\left[\frac{-12 \cdot i \cdot \left[i \cdot (n) \cdot \left(\pi - \frac{\Theta}{2} \right) \right]}{(n+1)^2} - \frac{-12 \cdot i \cdot \left[i \cdot (n) \cdot \left(\pi + \frac{\Theta}{2} \right) \right]}{(n+1)^2} \right] \cdot \left(\frac{i^{n-1} - i^{-n-1}}{2} \right) \right]$$



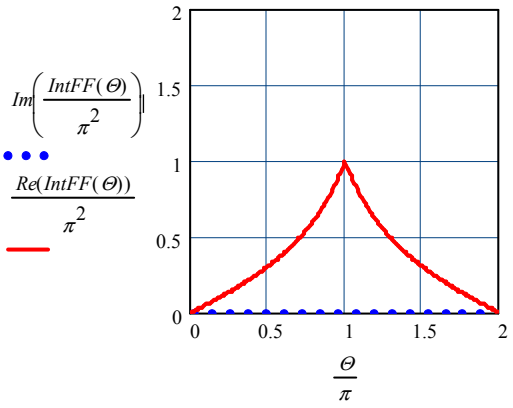


Figure 37 – Equation (IV.M) is not graphically equivalent over the range but it is graphically equivalent at the point $\Theta = \pi$.

Figure 38 – Calculations for the π^2 Equation (IV.P)

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

1. The integral on the right hand side of the GTE is:

$$IntFF_n := \left[\left[\frac{-12 \cdot i \cdot \left[i \cdot (n) \cdot \left(\pi - \frac{\pi}{2} \right) \right]}{(n+1)^2} - \frac{-12 \cdot i \cdot \left[i \cdot (n) \cdot \left(\pi + \frac{\pi}{2} \right) \right]}{(n+1)^2} \right] \cdot \left(\frac{i^{n-1} - i^{-n-1}}{2} \right) \right]$$

2. The value of this integral is:

$$\sum_{n=1}^{1000} \left[\left[\frac{-12 \cdot i \cdot \left[i \cdot n \cdot \left(\pi - \frac{\pi}{2} \right) \right]}{(n+1)^2} - \frac{-12 \cdot i \cdot \left[i \cdot n \cdot \left(\pi + \frac{\pi}{2} \right) \right]}{(n+1)^2} \right] \cdot \left(\frac{i^{n-1} - i^{-n-1}}{2} \right) \right] = 9.858 \qquad = \pi^2$$

3. The equation also provides the combination of harmonic components for each n:

$\frac{-12 \cdot i \cdot \left[i \cdot (n) \cdot \left(\pi - \frac{\pi}{2} \right) \right]}{(n+1)^2}$	$\frac{-12 \cdot i \cdot \left[i \cdot (n) \cdot \left(\pi + \frac{\pi}{2} \right) \right]}{(n+1)^2}$	$\left[\frac{-12 \cdot i \cdot \left[i \cdot (n) \cdot \left(\pi - \frac{\pi}{2} \right) \right]}{(n+1)^2} - \frac{-12 \cdot i \cdot \left[i \cdot (n) \cdot \left(\pi + \frac{\pi}{2} \right) \right]}{(n+1)^2} \right]$
3	-3	6
1.333i	1.333i	0
-0.75	0.75	-1.5
-0.48i	-0.48i	0
0.333	-0.333	0.667
0.245i	0.245i	0
-0.188	0.188	-0.375
-0.148i	-0.148i	0
0.12	-0.12	0.24

4. This combination can be written as a series:

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots.$$

5. This can be expressed in summation form:

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

6. The integral on the left hand side of the GTE = π^2 was factored out.

$\frac{i^{n-1} - i^{-n-1}}{2}$
1
0
-1
0
1
0
-1
0
1

$$\frac{IntFF_n}{6} =$$

1
0
0.25
0
0.111
0
0.063
0
0.04

Figure 38 shows the calculation and normalization. Expanding the series (IV.M) we obtain:

$$\frac{\pi^2}{12} = -i \cdot \left[\left(\frac{(+i)-(-i)}{2^2} \right) \cdot (+1) + \left(\frac{(-1)-(+1)}{3^2} \right) \cdot 0 + \left(\frac{(-i)-(+i)}{4^2} \right) \cdot (-1) + \left(\frac{(+1)-(-1)}{5^2} \right) \cdot 0.... \right]$$

$$= -2i \cdot \left[\left(\frac{2i}{4} \right) + 0 + \left(\frac{2i}{16} \right) + 0 + \left(\frac{2i}{36} \right) .... \right] \qquad = 2 \left[1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} .... \right] \tag{IV.N}$$

And we find our third series $$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ... = \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \tag{IV.O}$$

Since the sign change resulted in $\pi^2 / 12$ going to $\pi^2 / 6$ which is a difference of $\pi^2 / 12$, the value of the terms that were initially subtracted out and then added in is $\pi^2 / 24,$ so

$$\frac{\pi^2}{24} = \frac{1}{4} + \frac{1}{16} + \frac{1}{36} .... = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \tag{IV.P}$$

Since graphical equivalence was not achieved with (IV.M), this was not an equivalence condition. The study is incomplete and will be revisited. However with partial equivalence we were still able to generate the series which was our objective in this section.

Part V: A. Harmonic Analysis and the Nature of Space

Danny and I were in a paddle boat that we took out into the Bay. We were a tiny fish in a big sea feeling every wave out there from the tiniest ripple to the overpowering wake of the massive boats. The space we existed in was not flat, it was wavy. In our moment of insight we convinced ourselves that space itself is wavy and that waves should be the base unit to measure and construct with rather than yardsticks and atoms. It was the summer of '83 and such thoughts were a much needed escape for the insatiable hearts of the young men.

Harmonic Analysis does not prove that we exist in a world where the base unit is actually a wave. It is simply a mapping relation for modeling physical phenomena.

B. Conclusion

The fundamental constants *e* and π have irrational values but can be expressed as an infinite sum of rational numbers. Nature must have a method to provide these values so they can be used in natural phenomena. They cannot be left to chance.

This method has proven difficult to uncover. There are not enough spatial variables available to operate the mechanics needed to produce these constants. The same is true of other natural relations like the Fibonacci series. Our analysis suggests a shadow space that is virtual and unobservable. Natural constants and functions that need hidden variables rely on this virtual space for their underlying mechanics. The invisibility of this virtual space is precisely why these problems have gone unsolved.

There is an equivalence relation between real space and virtual space. Observers from both spaces must agree on the result of a common event. Therefore the results of mapping data in one space must agree with the results of mapping the same data in the other. Mathematically we would say that the integrals of the data maps over common boundaries are equal.

The equivalency of spatial mappings is not intuitive. We presented two examples that are easy to visualize. In the theatre problem the audience views complex motion in time space while the stage hand sees the same motion mapped in frequency space. The balloon problem is similar. Here the balloonist can only detect motion along one axis, while at the river it is apparent that wave action is causing motion in two dimensions. In both cases it is difficult for an observer to understand the alternate mapping. It is just not intuitive.

The base units in harmonic space are wave functions which are harmonically ordered, increasing in frequency and decreasing in amplitude. They can be thought of as vibration modes. There are an infinite number of these wave functions available, and, like a baking recipe, any amount of each individual component could be used. Therefore combining the wave functions to correctly map a natural phenomena requires a key or coding pattern dictating which components to use and how much of each. The component waves must combine in an exact way to produce a specific mode, somewhat analogous to an orchestra producing a specific sound.

The coding pattern is determined by the equivalence relation. The equivalence relation is a differential equation in integral form and has a solution set made from the base waves. The coefficients of this solution set are the coding pattern. They indicate which component waves to use and how much of each are necessary in order to meet the requirement that the equivalence relation demands: the results of both spatial maps must agree.

The coding pattern is a rule set. It requires the waves to combine in a certain way that results in equivalence with a physical action. That is how we determined the harmonic functions, through graphical equivalence.

The inverse is true as well. The physical action must equally comply with the rules that govern this harmonic combination. That is how nature provides the missing information for the fundamental constants and natural equations. The harmonic waves themselves are not observable in real space. It is their coding patterns required for equivalence which determines the values we observe.

The harmonic combination rules are precisely the type of rule sets that nature needs to control certain natural phenomena, but these rules are unavailable with the limited selection of real space variables. Instead there is an equivalence relation marrying real space to this harmonic space. Harmonic space is complex and needs combination rules to map an action. In doing so the virtual system provides the missing information in a seamless and invisible way.

This is nature's method of storing hidden values and implementing them. It works similar to a guitar string. The guitar string can only make certain tones. These are determined by a differential equation involving the physical properties of the string and its end conditions. If we build another string using the same equation we will get the same note. The string and its equation store information (the note). When it is plucked the string must obey the differential equation, and implementing this equation releases the stored information. The equivalence relation works in the same way. The value of π for example is stored in a differential equation involving the physical properties of the circle and its boundary conditions. When an action involving π occurs the solution of the differential equation (GTE) is called upon and the stored information is released. This requires the physical process to occur in a specific way analogous to the guitar string vibrating in a specific mode. So the value of π is not a geometry problem after all. It is a coding problem. The value of π is stored in the harmonic coding pattern that satisfies the equation. It is there for all to use for all of time just like the rules governing musical instruments.

When we examined the Euler Equation we saw that it was an example of a single spinner. In the π analysis we extended this model to two synchronized spinners. This enables control of two complex numbers with one variable, and the difference between these values provides the code. The Fibonacci Series is similar. Here two numbers are also controlled by one variable. They contain the information on the current value and the next value for each cycle, and therefore encode the value at every cycle in a single equation.

Nature relies on these coding patterns and there is an analysis method available to decode them. In doing so we can provide a better understanding of the underlying mechanics of natural phenomena.

C. Ubiquity

A single equation, the GTE, was used for determining e, π and the Fibonacci numbers. In general it provides a method for nature to encode and to store additional variables when insufficient real variables exist for natural phenomena. It also provides a transformation relation so that one set of data can imply the existence of another set of data.

We expect this work to have applications in many other fields.

D. Dedication

This work is dedicated in memory of our fallen brother and mentor Mike Chavez. The general transformation equation shall be known as The Chavez Equation.

E. References

1. What Makes an Equation Beautiful, The New York Times, October 24, 2004

Part VI: A. Fibonacci Numbers

Here we derive the equation for the Fibonacci numbers and show how the mechanics works. It is assumed that the reader is familiar with the work on e and π so this discussion streamlined. The equation we will derive provides the n[th] Fibonacci Number f_n :

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} \tag{60}$$

Each Fibonacci number is an integer and is the sum of the two Fibonacci numbers before it. Here nature is tasked with tracking two numbers and adding them together. Let us see how this is done.

We start by measuring an action in both real and harmonic space and require that the results be the same for both observers. This is our Equivalence Relation or General Transform Equation (GTE):

GTE $$\int_A^B f(x)dx = \int_A^B F(X)dX \tag{6.1}$$

We let $f(x) = f_n$, where f_n is the n[th] Fibonacci number. We seek F(X) which satisfies Equation (6.1).

Let $F(X) = n \cdot X^{n-1}$ Then $$\int_A^B f_n \cdot dx = \int_A^B n \cdot X^{n-1} dX \tag{6.2}$$

The choice of the boundary conditions A and B is very important as this is what governs the tracking numbers for the series. *This implies that we could track other series with rule sets different than Fibonacci by using different values for A and B.* For now we state

$$A = (1-\sqrt{5})/2 \quad \text{and} \quad B = (1+\sqrt{5})/2 \tag{6.3}$$

which we will address later. So

$$\int_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} f_n dx = \int_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} n \cdot X^{n-1} dX \tag{6.4}$$

Evaluating the left hand side of (6.4):

$$\int_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} f_n dx = f_n \cdot x\Big|_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} = f_n \cdot \left(\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right) = \sqrt{5} f_n \tag{6.5}$$

Evaluating the right hand side of (6.4):

$$\int_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} n \cdot X^{n-1} dX = \quad X^n \Big|_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} \tag{6.6}$$

Substituting (6.5) & (6.6) into (6.4):

$$\sqrt{5} f_n = X^n \Big|_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} \tag{6.7}$$

Therefore the n[th] Fibonacci integer must equal:

$$f_n = \quad \frac{1}{\sqrt{5}} \cdot \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right] = \quad \frac{\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n}{\sqrt{5}} \tag{6.8}$$

This is our desired result and we have completed the derivation of (6.0), with exception of the Boundary Conditions. Now we want to look at the mechanics of this.

B. Equivalence

The derivation of (6.8) came from the GTE using a similar procedure to that of e and π . Also, since each Fibonacci integer is the sum of the two Fibonacci integers before it, we can write

$$\int_A^B f_n \cdot dx = \quad \int_A^B F(X) dX = \quad \int_A^B F_{n-1}(X) dX + \int_A^B F_{n-2}(X) dX \tag{6.9}$$

where F_{n-1} and F_{n-2} are the harmonic transforms of the previous Fibonacci integer and the one before that, respectively.

When an action occurs in real space that requires the Fibonacci Numbers it is demanding a solution to Equation (6.9) which is a differential equation (DE) expressed in integral form. This is analogous to plucking a guitar string. The guitar string can only vibrate in certain modes determined by the DE. The action demands a solution to the DE and is subject to its mechanics. This is the method of implementation.

In this case the n^{th} Fibonacci Number, f_n , equals its harmonic transform per (6.8) so there is no need to show equivalence between these. f_n is determined by the difference of $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$ raised to increasing powers of n.

According to (6.9) f_n is equal to the sum of the integrated harmonic transforms of the two previous numbers F_{n-1} and F_{n-2}. This implies:

$$F_{n-1} = (n-1)\cdot X^{n-2} \quad \text{and} \quad F_{n-2} = (n-2)\cdot X^{n-3} \tag{6.10}$$

To satisfy (6.9) we let:

$$\sqrt{5} f_n = X^n \Big|_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} = X^{n-1} \Big|_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} + X^{n-2} \Big|_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} \tag{6.11}$$

And these are the two previous Fibonacci integers:

$$f_{n-1} = \frac{1}{\sqrt{5}} X^{n-1} \Big|_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} \qquad f_{n-2} = \frac{1}{\sqrt{5}} X^{n-2} \Big|_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} \tag{6.12}$$

So the harmonic transforms add in the same way that the Fibonacci Numbers do.

And in general the n[th] Fibonacci number is given by:

$$f_n = \frac{1}{\sqrt{5}} X^n \Big|_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} \tag{6.13}$$

Equation (6.11) is used to demonstrate graphical equivalence. The n^{th} Fibonacci Number, f_n is equal to the sum of the integrated harmonic transforms of the two previous numbers F_{n-1} and F_{n-2}. See Figure 60.

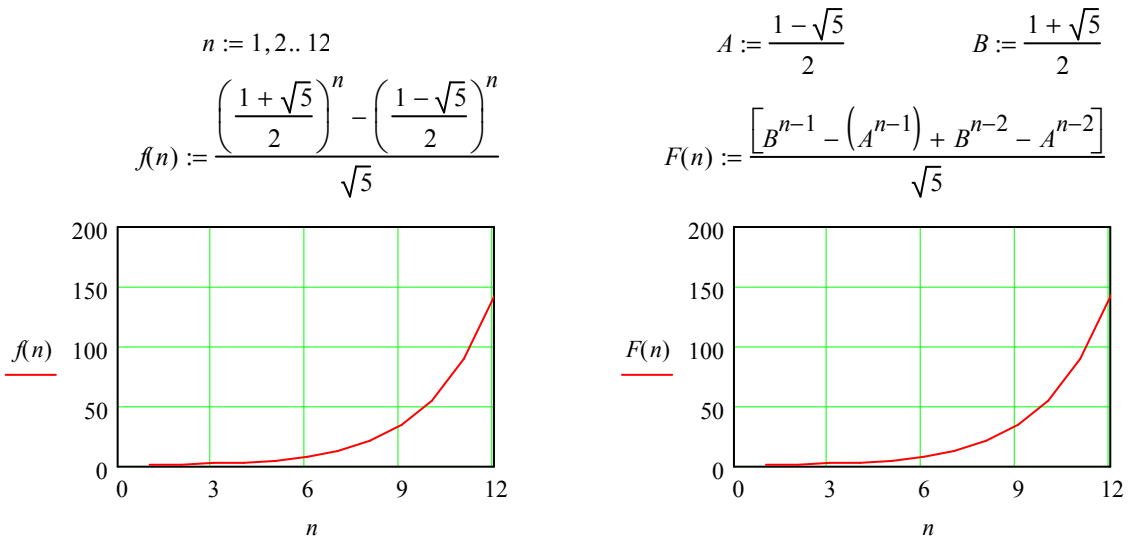


Figure 60 - The n^{th} Fibonacci Number, f_n is equal to the sum of the integrated harmonic transforms of the two previous numbers F_{n-1} and F_{n-2} as per Equation (6.11).

C. Binomial Expansion & Tracking Numbers

We now look at how the boundary conditions A and B govern the mechanics of the differential equation,

enabling it to satisfy the requirement that the nth Fibonacci number is the sum of the previous two values, and also that all of these values are integers.

Equation (6.13) is a remarkable equation. Not only does it tell us what the nth Fibonacci number is, it also tells us and what to add to this to get the next Fibonacci number. It does this by tracking two numbers. One determines the current value and the other what to add to get the next value. It is able to do this by raising the mixed numbers $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$ to increasing powers of *n*. Under binomial expansion the rational and irrational portions combine separately, and there are two coefficients, one for the rational portion and one for the irrational portion. These are the two tracking numbers. When (6.13) requires us to take the difference of the two expansions, the irrational portions cancel out and we get a rational (integer) answer. In this way one of the tracking numbers is hidden and unobservable. This is another instance where there are insufficient variables in real space for the natural process. Here Nature needed a way to track a second value. The DE (6.9) is the method and the solution (6.13) contains the coding pattern. *In general the presence of rational and irrational numbers in a DE may indicate that unobservable values are being tracked.*

The information provided by (6.13) is a complete set meaning that by tracking these two numbers for each n, the entire set is determined. The evaluation at $n-1$ tells what needs to be added to reach that at n. That is equivalent to what needs to be subtracted from the value at n to get the value at $n-1$.

Now we examine the underlying mechanics. We return to Equation (6.0)

$$f_n = \frac{1}{\sqrt{5}} \cdot \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right] \tag{6.0}$$

At each *n*, there is a binomial expansion of the complex terms. For $n=2$:

$$\left(\frac{1+1\sqrt{5}}{2} \right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{3+1\sqrt{5}}{2} \qquad \text{and} \qquad \left(\frac{1-1\sqrt{5}}{2} \right)^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{3-1\sqrt{5}}{2} \tag{6.14}$$

The real and irrational parts of the expansion do not mix. The real numbers add to real numbers and the irrational numbers add to irrational numbers. In this way two separate properties can be controlled by a single mixed number.

As we will show, only one of these numbers is physically observable. The unobservable number is "dormant" in the physical process. Note that the binomial expansion results in a term where the irrational component of base $\sqrt{5}$ multiplies by another component of base $\sqrt{5}$ to produce a real number of base 5 which then adds in to the real numbers. In this manner the unobservable or dormant term plays a part in the physical process but is not visible. It contains information on how to calculate the value at future cycles where each n is a cycle. Figure 61 shows the binomial expansion for $n=1$ to $n=6$. Note that for each n there are two separate coefficients, one for the rational portion and one for the irrational portion. Such a property is imperative to natural phenomena such as growth. A plant may currently have three leaves, but that condition must also contain the information on the number of leaves in the next growth cycle, and that one must contain the information for the one which follows.

n=1	$\left(\frac{1+\sqrt{5}}{2}\right)^1$	$=$	$\frac{1+\sqrt{5}}{2}$	$\left(\frac{1-\sqrt{5}}{2}\right)^1$	$=$ $\frac{1-\sqrt{5}}{2}$
n=2	$\left(\frac{1+\sqrt{5}}{2}\right)\cdot\left(\frac{1+\sqrt{5}}{2}\right)$	$=$	$\frac{3+\sqrt{5}}{2}$	$\left(\frac{1-\sqrt{5}}{2}\right)\cdot\left(\frac{1-\sqrt{5}}{2}\right)$	$=$ $\frac{3-\sqrt{5}}{2}$
n=3	$\left(\frac{1+\sqrt{5}}{2}\right)\cdot\left(\frac{1+\sqrt{5}}{2}\right)\cdot\left(\frac{1+\sqrt{5}}{2}\right)$	$=$	$\frac{4+2\sqrt{5}}{2}$	$\left(\frac{1-\sqrt{5}}{2}\right)\cdot\left(\frac{1-\sqrt{5}}{2}\right)\cdot\left(\frac{1-\sqrt{5}}{2}\right)$	$=$ $\frac{4-2\sqrt{5}}{2}$
n=4	$\left(\frac{1+\sqrt{5}}{2}\right)^4$	$=$	$\frac{7+3\sqrt{5}}{2}$	$\left(\frac{1-\sqrt{5}}{2}\right)^4$	$=$ $\frac{7-3\sqrt{5}}{2}$
n=5	$\left(\frac{1+\sqrt{5}}{2}\right)^5$	$=$	$\frac{11+5\sqrt{5}}{2}$	$\left(\frac{1-\sqrt{5}}{2}\right)^5$	$=$ $\frac{11-5\sqrt{5}}{2}$
n=6	$\left(\frac{1+\sqrt{5}}{2}\right)^6$	$=$	$\frac{18+8\sqrt{5}}{2}$	$\left(\frac{1-\sqrt{5}}{2}\right)^6$	$=$ $\frac{18-8\sqrt{5}}{2}$

Figure 61 – Binomial Expansion used for the Fibonacci Numbers. Note there are two coefficients for each n, one for the rational part and one for the irrational part.

$\underset{\downarrow}{t_u}\ \underset{\downarrow}{t_o}\quad \underset{\downarrow}{t_u}\ \underset{\downarrow}{t_o}\quad \underset{\downarrow}{f_n}$

n=1	$\frac{1}{\sqrt{5}}\cdot\left(\frac{1+1\sqrt{5}}{2}-\frac{1-1\sqrt{5}}{2}\right)=1$
n=2	$\frac{1}{\sqrt{5}}\cdot\left(\frac{3+1\sqrt{5}}{2}-\frac{3-1\sqrt{5}}{2}\right)=1$
n=3	$\frac{1}{\sqrt{5}}\cdot\left(\frac{4+2\sqrt{5}}{2}-\frac{4-2\sqrt{5}}{2}\right)=2$
n=4	$\frac{1}{\sqrt{5}}\cdot\left(\frac{7+3\sqrt{5}}{2}-\frac{7-3\sqrt{5}}{2}\right)=3$
n=5	$\frac{1}{\sqrt{5}}\cdot\left(\frac{11+5\sqrt{5}}{2}-\frac{11-5\sqrt{5}}{2}\right)=5$
n=6	$\frac{1}{\sqrt{5}}\cdot\left(\frac{18+8\sqrt{5}}{2}-\frac{18-8\sqrt{5}}{2}\right)=8$

Figure 62: Coding of the Fibonacci Numbers. t_u subtracts out to zero but determines the next f_n.

Figure 62 contains the calculation for the first six Fibonacci Numbers. The first thing we note is that the function produces the correct Fibonacci integer f_n.

We also see the rational and irrational parts have two tracking integers t_u and t_o. t_o is observable and provides the value of the Fibonacci Number f_n. t_u is unobservable and subtracts out to zero. From Figure 61, for $n=3$, $t_o=2$ and $t_u=4$, and separating the rational and irrational parts we get the results in Fig 62:

$$\frac{1}{\sqrt{5}}\cdot\left(\frac{2\sqrt{5}}{2}-\frac{-2\sqrt{5}}{2}\right)=2=f_3 \quad \text{and} \quad \frac{1}{\sqrt{5}}\cdot\left(\frac{4}{2}-\frac{4}{2}\right)=0$$

Although t_u subtracts out and is unobservable it is still important. It is needed to determine the next set of values. If we add t_u to t_o and divide by 2 we get the next Fibonacci number. For $n=3$: $\frac{4+2}{2}=3=f_4$.

In general: $$f_{n+1}=\frac{t_u+t_o}{2} \qquad (6.15)$$

In this way nature stores the information for future cycles in the current cycle through binomial expansion of a mixed rational and irrational number. In (6.4) we required the GTE be true at these boundary conditions. (6.13) is the solution to this equation and uses the difference of the two expansions which results in t_u dropping out. Yet it is necessary for it to be in the solution set in order to solve the DE.

The tracking numbers are added to each cycle. Every time we raise n to the next integer that is equivalent to multiplying the n^{th} base by $(1\pm\sqrt{5})/2$ one more time. Referring to $n=3$ and $n=4$, from Figure 61 we see that the fourth power expansion equals the third power times the first power:

$$\left(\frac{1+1\sqrt{5}}{2}\right)^4=\left(\frac{1+1\sqrt{5}}{2}\right)^3\cdot\left(\frac{1+1\sqrt{5}}{2}\right)^1$$

$$=\left(\frac{4+2\sqrt{5}}{2}\right)\cdot\left(\frac{1+1\sqrt{5}}{2}\right)$$

$$=\left(\frac{4+2\sqrt{5}+4\sqrt{5}+10}{4}\right)=\left(\frac{7+3\sqrt{5}}{2}\right) \tag{6.16}$$

And in general $$\left(\frac{1+1\sqrt{5}}{2}\right)^n=\left(\frac{1+1\sqrt{5}}{2}\right)^{n-1}\cdot\left(\frac{1+1\sqrt{5}}{2}\right)^1 \tag{6.17}$$

This is not a special property for this particular number, it applies to all numbers. What is special about this number is that each tracking integer is the sum of the two before it. So not only does the Fibonacci number follow this rule, the tracking numbers do as well. So in addition to the previous equation:

$$\left(\frac{1+1\sqrt{5}}{2}\right)^n=\left(\frac{1+1\sqrt{5}}{2}\right)^{n-1}+\left(\frac{1+1\sqrt{5}}{2}\right)^{n-2} \tag{6.18}$$

D. Determination of Boundary Conditions

We have thus far used $(1\pm\sqrt{5})/2$ as the boundary conditions without explanation. The operation in Equation (6.17) could have applied to expanding any number. That of (6.18) is specific to the Fibonacci numbers as it requires the current value to be equal to the sum of the two previous values. $(1\pm\sqrt{5})/2$ are the special numbers that work in Equation (6.18). If they were unknown we could solve for them. Let y be the special number. Then (6.18) becomes:

$$(y)^n=(y)^{n-1}+(y)^{n-2} \tag{6.19}$$

The equation is true for all n so we can choose any value. If we let $n=2$ we get $y^2=y^1+y^0$ or

$$y^2-y-1=0 \tag{6.20}$$

The solutions to this equation are of course $y = (1 \pm \sqrt{5}) / 2$. (6.21)

Therefore the boundary conditions A and B are selected as $(1 \pm \sqrt{5}) / 2$ to satisfy $y^n = y^{n-1} + y^{n-2}$ which is the Fibonacci relation, add the previous two values to get the next.

E. The Golden Ratio

The Fibonacci numbers are associated with the Golden Ratio and the infinite rectangle pattern. Here we have an outer rectangle with sides y and x with an infinite number of inset rectangles of the same proportion. The ratio of the sides of outermost rectangle is $y : x$, and for the first inset rectangle $x : (y - x)$. If these two rectangles are of the same proportion then $y / x \;=\; x / (y - x)$.

This is true for all x. If we let $x = 1$ this reduces to $y = 1 / (y - 1)$ which equals $y^2 - y - 1 = 0$. This is the same as Equation (6.20) with solution $y = (1 \pm \sqrt{5}) / 2$. Only the positive solution is applicable in this case. So the golden ration is:

$$(1 + \sqrt{5}) / 2 \tag{6.22}$$

The presence of the irrational component $\sqrt{5}$ indicates the problem is more complex than real space allows us to observe. When natural phenomena follow Fibonacci Numbers or the Golden Ratio they are following the rules set forth by the Equivalence Relation or GTE. The solution set contains an unobservable tracking number used to determine the next value. In this manner the entire series is encoded in a single equation. The key to the operation is the boundary conditions in the Equivalence Relation. These are determined by requiring that the value at any cycle be equal to the sum of the previous two cycles, which is simply the definition of the Fibonacci Series. Thus the GTE derives the Fibonacci Equation from first principles.

It also provides a means to implement this. The GTE sets requirements for a specific action analogous to the equations that determine the allowable modes of a guitar string. When the action occurs it must conform to the rules of a differential equation and there are specific allowable states. Here there is a relationship of addition between cycles and this dictates what states, or Fibonacci Numbers, the differential equation will allow.