Part III: Pi 1. Introduction

In this section we show how the value of π can be determined using the General Transformation Equation or GTE. The GTE is the same general formula used for determining the value of e however here we will use a polar coordinate system instead of Cartesian.

Like e, π is a fundamental constant of nature which has an irrational value and can be expressed as an infinite sum. We will derive the famous Gregory Series for π:

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \qquad \text{(III.iA)}$$

Additionally we show the mechanics of how the equation works through a model of two opposing spinners which produce the natural code.

We must have a physical action. Suppose one takes a measurement around the outside of a pipe with a measuring tape. As the tape is laid down more of the circumference is measured, first one inch, then two, then three, etc. Although this occurs on a surface it can also be seen as a one dimensional operation because we can mark the tape, then unroll it, and view our measurements in flat space on a line. Not only is it one dimensional, it has no richness of structure. The arc length is proportional to the angle subtended. With $r = 1$ the arc length and angle are equal. See Figure 20.

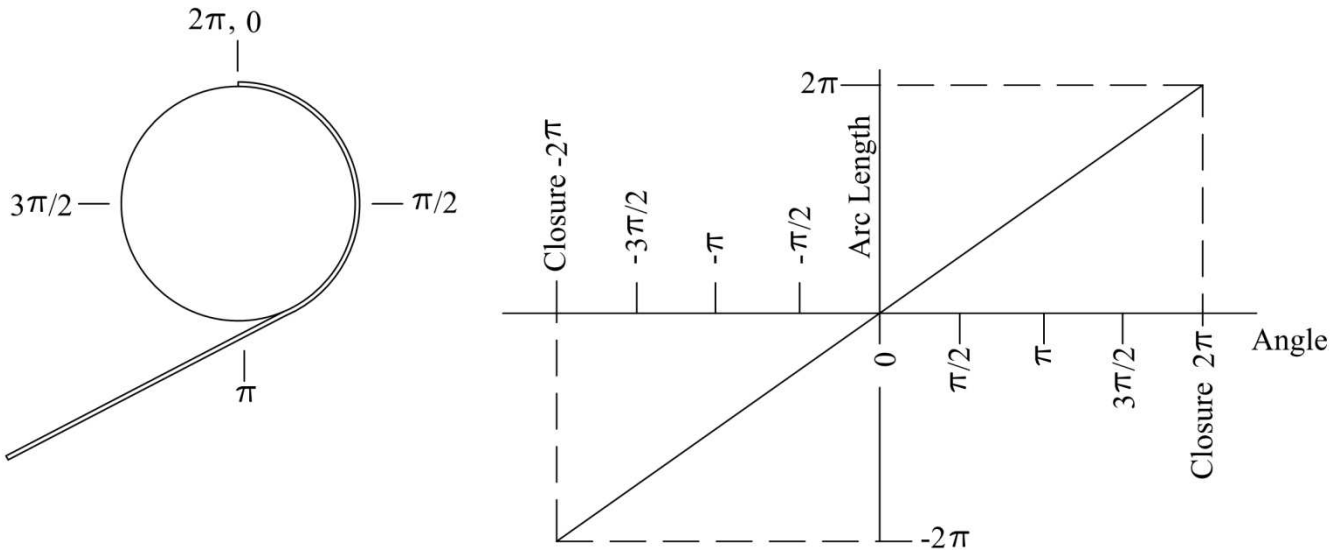


Figure 20 – At left is the measurement of a pipe with a measuring tape. At right the tape is unrolled and the arc length is graphed vs. the angle measured. The surface is two dimensional but the arc length exists only along the circumference and can be mapped in C Space as shown. It is also possible to measure in the other direction and obtain a negative value. The harmonic space model must replicate this graph.

In the Spring Theatre we had a mechanical system behind a screen that had rich structure but the audience could not see it. They could only see the light attached to the mechanism and could not explain its behavior. In this section we will create an equation which graphs identical to the circumferential measurement operation

of Figure 20. The measurement operation occurs in real space. The one we create is mapped in a virtual space composed of harmonic elements which are unit-less and cannot be observed. The reason why we do this is that the harmonic space is rich in structure and because of this there are rules governing the mechanics.

The rules require that the harmonics be combined in a specific pattern in order to equate with the measurement operation in real space. That pattern determines the value of π.

The reason for this is that the equation in harmonic space models the real space equation exactly. They both map a common action but on different types of maps. Since this is a real action there is an equivalence relation that requires that the results observed in both spaces must agree. This equivalence relation provides a numerical relationship between the two spatial maps and they are said to be transformations of one another. The equation for relating these two maps for a single action common to both spaces is described by the General Transformation Equation or GTE. The left hand side represents the action $A \rightarrow B$ in real space and the right hand side represents mapping the same action in harmonic space:

GTE
$$\int_A^B f(\theta)d\theta = \int_A^B F(\Theta)d\Theta \qquad \text{(III.iB)}$$

The two functions $\int f(\theta)d\theta$ in real space and $\int F(\Theta)d\Theta$ in harmonic space are connected. When the measurement operation occurs in real space the rule operations and subsequent mechanics are operating in harmonic space. They are inseparable, like two machines that are on the same shaft and must run together, but are located in different rooms. If one machine is geared to move in increments, both machines must move this way since they are on the same shaft. So when $f(\theta)$ moves an angle $d\theta$ in real space this corresponds to $F(\Theta)$ moving by the same amount when measured relative to $d\Theta$ in harmonic space.

For equivalence, the individual harmonic wave functions must combine in a specific way to form the function $\int F(\Theta)d\Theta$ which maps identical to $\int f(\theta)d\theta$ in real space. Since both are observations of the same event equivalence must exist. The specific harmonic wave combination is a code that invokes rules about how the action can occur. Like the two machines on the same shaft, the two mapping functions are connected and the rules governing actions in harmonic space are effect the actions in real space. In this way the remnants of the harmonic combination rules are detectable in real space. These combination rules determine the value of π. π must have this value. Otherwise the GTE would be violated and the two maps describing the same action would show different results.

Virtual But Describable

We can't see harmonic space but we can describe it mathematically. We can develop equations and rules on paper and see how they work even though we can't observe the space itself.

Harmonic space is wavy and the base units are wave functions or harmonics. The wave functions are numbered and get smaller in amplitude but increase in frequency over the set. There are an infinite number of these harmonic bases available, but since they decrease in amplitude the effect of using additional terms diminishes as the harmonic number n increases. So as $n \rightarrow \infty$, $\int F(\Theta)d\Theta \rightarrow \int f(\theta)d\theta$.

In order to produce the specific function there is a rule set or coding pattern which determines which waves are added, subtracted, or not used. We will see that the rule set for our measurement requires that

$$\frac{\pi}{4} = 1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \frac{1}{7} + 0 - \frac{1}{9} + 0 + \ldots \qquad \text{(III.iC)}$$

or in summation notation

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \qquad \text{(III.iD)}$$

This is the formula we will derive. It is the famous Gregory Series for π which has always been mysterious. The equivalence relation requires that the results of the mappings in the two spaces agree. The consequence of this is that π must have this value.

It is astounding that the component waves can add up to model this fundamental constant. It seems to suggest that harmonic space is more natural for mapping π than real space. Even more astounding is that there is a specific pattern for adding the components and this pattern is encoded in the GTE.

Equally amazing is the way the equation works. When we calculate the left side of the GTE, $\int f(\theta)d\theta$ over the boundary values, we will get a multiple of π. When we calculate the right side, $\int F(\Theta)d\Theta$, all vestiges

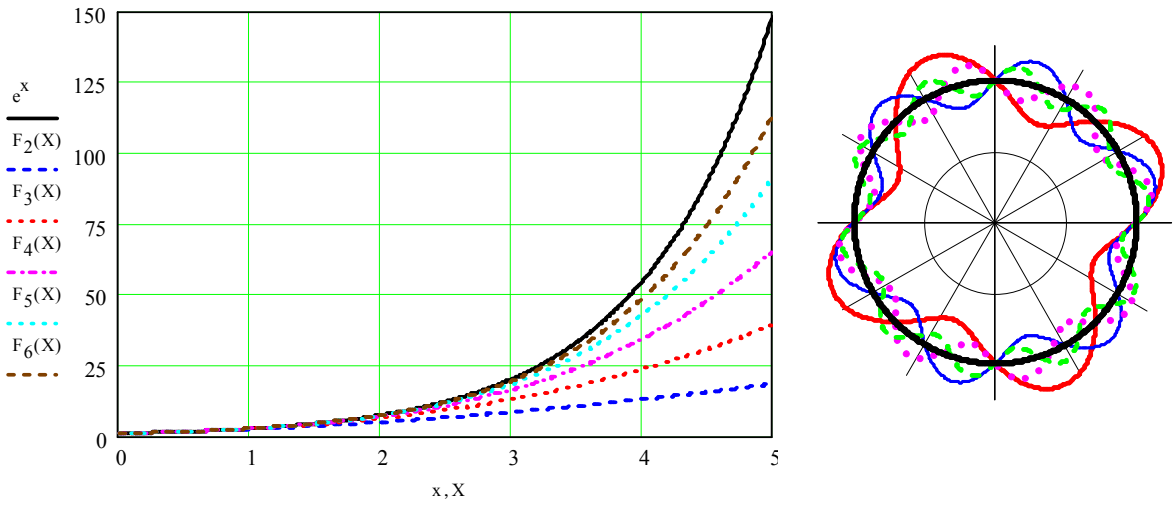


Figure 21 – (Left): In the analysis for *e* we added the wave forms to create a function that models e. The traces shows progressive summing of modes 2 thru 6 and the sum of all modes equals *e*. (Right): A model for visualizing the harmonic modes that sum to form $F(\Theta)$ in the analysis of π. Four of the harmonic components are shown. The harmonic modes are in some ways analogous to vibration modes of a ring.

of the harmonic space disappear and only the rules of how to combine the base waves remain, but not the waves themselves. This establishes an equation relating π and the combination rules in harmonic space.

Thus the transformation equation requires that harmonic space be unobservable and that π be determined by the harmonic combination rules. We saw this occur with *e.* The base waves dropped out of the equation requiring *e* to be equal to the sum of the eigenvalues of the differential equation. The sum of harmonics is used to model both *e* and π (See Figure 21). However in the case of π the harmonics must add up to agree with the results of the measurement taken along the circumference of a circle. We can imagine the harmonic modes as perturbations or vibration modes of a ring, which must add up to form the ring and must exhibit properties of continuity and closure.

In our analysis we will always take the radius of the circle to be $r = 1$. We consider the circle as a base unit and 2π is the angle turned which results in closure. With $r = 1$ the arc length C that is measured has the same value as the angle turned, θ. This is because $C = \int r \cdot d\theta = \theta$. To distinguish between the arc length C and angle turned θ, we will refer to arc length in C space and angle in θ space. See Figure 22 *top left*.

With $r = 1$ we can usually use arc length and angle interchangeably since they have the same numerical value. However it is important to recognize that arc length exists in C space. It is a length measurement confined to the circumference of a circle and is a practical way to take physical measurements. We can unroll this into a Cartesian graph and the function is then mapped onto a straight line (Figure 20). When we use the GTE it is more correct to use θ space or angle turned because harmonic space is unit-less while C space would have some length unit attached. We could overcome this dilemma by referring to the C / r ratio however we feel it is simpler to work with the one number θ.

Figure 22 *top right* is a drawing to help visualize the harmonic modes. They are analogous to vibration modes of ring. These waves are thought of as perturbations of the circumference and they have a special relationship to it. The correct combination of these waves of the circumference sums to equal the circumference.

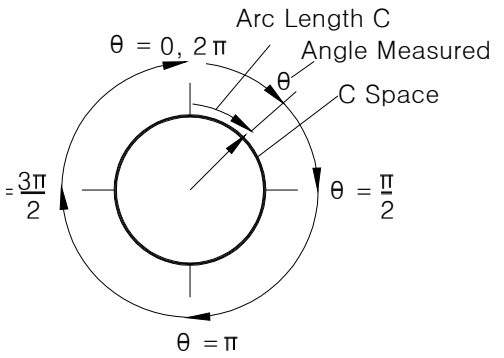
Figure 22 *bottom* graphs the first four harmonics. Note the increasing frequency and decreasing amplitude. The sum of the first four harmonics and the first 100 harmonics are also shown. Note how the sum of the harmonics is trending toward the real measurement; that is the linear graph shown in Figure 20. The relationship gets better with increasing n and is exact when $n = \infty$.

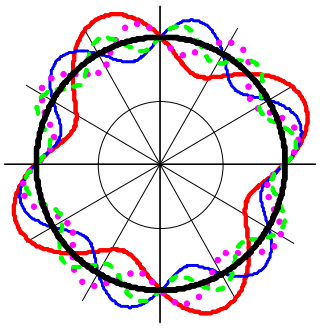
The integrals F_n for the first four harmonics on the graph are based on the formula:

$$\int F_n(\Theta) d\Theta = \frac{2}{n \cdot i} \cdot \left[e^{in\left(-\frac{\theta_m}{2}+\pi\right)} - e^{in\left(\frac{\theta_m}{2}+\pi\right)} \right] \qquad \text{(III.iE)}$$

The summation of the first four harmonics is below. The 100 summation is similar.

$$SumF_4 = \sum_{n=1}^{4} \int F_n(\Theta) d\Theta = \sum_{n=1}^{4} \frac{2}{n \cdot i} \cdot \left[e^{in\left(-\frac{\theta_m}{2}+\pi\right)} - e^{in\left(\frac{\theta_m}{2}+\pi\right)} \right] \qquad \text{(III.iF)}$$





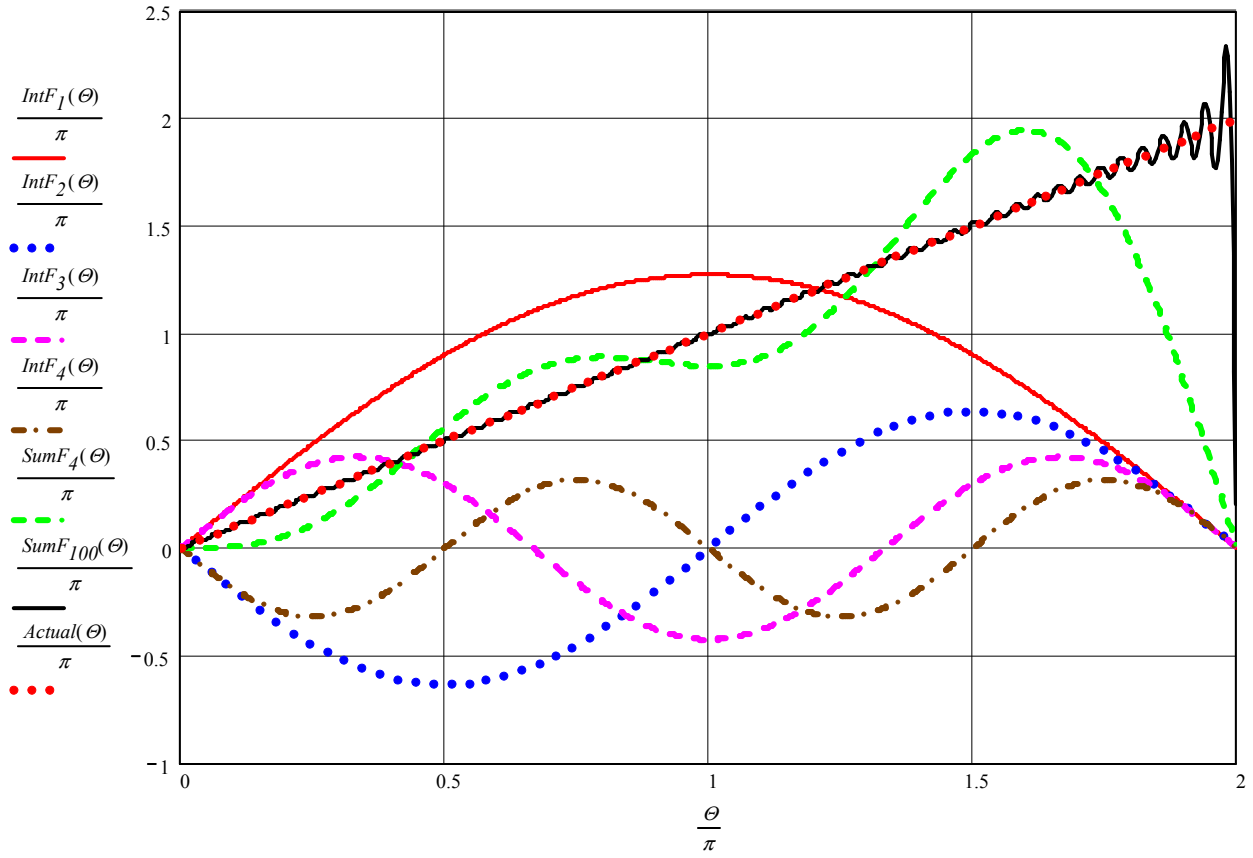


Figure 22 – *Top Left*: the arc length C is confined to the circumference and equals the angle turned θ when $r = 1$. *Top Right*: Some of the harmonic modes. *Bottom*: The circle is unrolled into a flat line with the first four harmonics plotted. The sum of the first 4 and first 100 harmonics is also shown. Note how at $n = 100$ the sum of the harmonics is trending toward the linear graph from of the actual measurement from Figure 20. The relationship is exact at $n = \infty$. The graphed equations are based on the formula:

$$\int F_n(\Theta)\,d\Theta = \frac{2}{n \cdot i} \cdot \left[e^{in\left(-\frac{\theta_m}{2}+\pi\right)} - e^{in\left(\frac{\theta_m}{2}+\pi\right)} \right]$$

2. Definitions

We define the angle which creates closure for the circle to be equal to 2π, and by extension a semicircle must equal π. While this may seem like we put π into the equation and we get π out, if one examines the mathematics it is apparent that this is not the case. We simply need a symbol to represent closure of the circle in real space and that is 2π. We do not assign a value to this. It must equal the sum of the harmonic components needed to reproduce it because of the GTE.

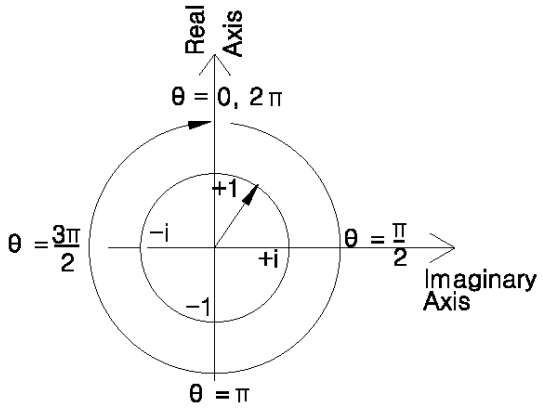


Figure 23

As part of this definition for π, we require that the circle have closure. That is to say that

$e^{i2\pi} = +1$, and by extension $e^{i\pi} = -1$, $e^{i\pi/2} = +i$, and $e^{i3\pi/2} = -i$. See Figure 23.

We stress here that we are not giving π a value. We are merely defining 2π as the angle θ which causes $e^{i\theta}$ to close. In this respect the Euler equation is simply a map laying out North, South, East and West and giving values 1, -1, $+i$ and $-i$ to define turns relative to these. The Euler equation does not give us the value of π. This is discussed in greater detail in the earlier discussion of the Euler equation.

Note that Section 4 which contains an overview of the MH equations is in many ways complete on its own. Advanced readers may wish to skip Section 3.

3. Development of Equations & Normalization

Here we develop the harmonic equations and normalization and demonstrate how the mechanics works. Let $f(\theta)$ be the function in real space for determining arc length along a circle of $r = 1$ with θ being the angle turned. Let $F(\Theta)$ be the corresponding function in harmonic space that we must determine. There is an equivalence relation between the real space and harmonic space maps of the action $A \rightarrow B$ called the General Transform Equation or GTE:

GTE
$$\int_A^B f(\theta)d\theta = \int_A^B F(\Theta)d\Theta \qquad \text{(III.A)}$$

We are looking for the function $F(\Theta)$ which solves the GTE. The integral of that function in harmonic space must have the same value as the integral of $f(\theta)$ in real space at every point on the interval. The general form of the right hand side of (III.A) is:

$$F(\Theta) = \sum_{n=0}^{\infty} a \cdot e^{in\cdot(b\Theta+c)} \qquad \text{(III.B)}$$

$F(\Theta)$ is the sum of a set of n wave functions of variable Θ and a, b, and c are normalization constants. If we imagine the circle to be a ring of solid material we can visualize the harmonic modes as vibration modes of the ring, in which the component waves must add up to create the circle in real space. See Figure 24. That is what (III.A) states mathematically.

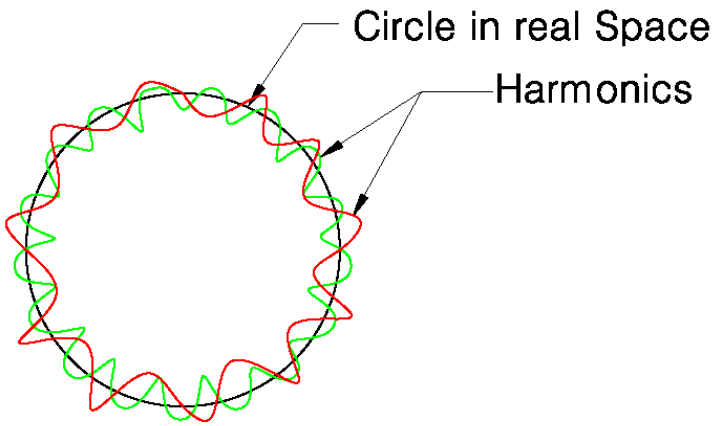


Figure 24

The harmonic modes can be thought of as sinusoidal vibrations of a ring. The harmonics sum to create a function that when integrated models π exactly.

We stress this is only an analogy. The circle does not really exhibit vibrations as the harmonics take place in virtual space.

Looking at the left side of (III.A) we see that $f(\theta)=1$ since the left side represents the action of measuring the arc length C. After all, $C=\int r\cdot d\theta$ and $r=1$.

And the GTE (III.A) can be written as

$$\int_A^B 1\cdot d\theta=\int_A^B\sum_{n=0}^{\infty}a\cdot e^{in\cdot(b\Theta+c)}d\Theta \qquad \text{(III.C)}$$

We will now make a first guess at $F(\Theta)$ ***and*** $f(\theta)$, ***then correct our assumptions based on the result.***
For our first guess we try $b=1,\quad a=2,$ and $c=0.$

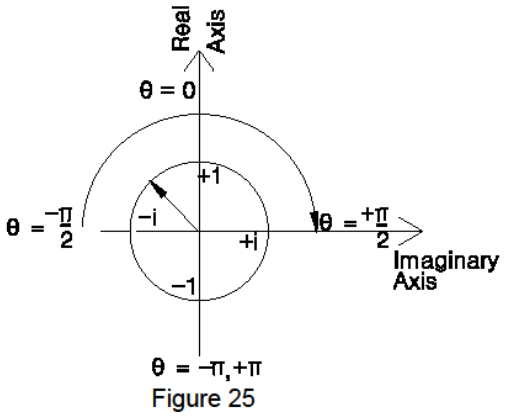


Figure 25
The boundaries of integration are -π/2 and +π/2 which is a semicircle in θ Space

Therefore our first guess is: $f(\theta)=1$ and $F(\Theta)=\sum_{n=0}^{\infty}2\cdot e^{in\cdot\Theta}$ (III.D)

Let θ_m be the specific angle measured. Then the boundary values *A* and *B* are given by $-\frac{1}{2}\theta_m$ and $+\frac{1}{2}\theta_m$ respectively. See Figure 25. We do not use 0 as a boundary of integration as this would not result in a solution to the GTE. In general harmonic analysis requires us to use negative boundaries in symmetry with positive ones.

For the left side of the GTE

$$\int_{-\theta_m/2}^{+\theta_m/2}f(\theta)d\theta=\int_{-\theta_m/2}^{+\theta_m/2}1\cdot d\theta=\theta\Big|_{-\theta_m/2}^{+\theta_m/2}=\theta_m \qquad \text{(III.E)}$$

For the right side of the GTE

$$\int_{-\theta_m/2}^{+\theta_m/2}\mathrm{F}(\Theta)d\Theta=\int_{-\theta_m/2}^{+\theta_m/2}\sum_{n=0}^{\infty}2\cdot e^{in\cdot\Theta}d\Theta=\quad\sum_{n=0}^{\infty}\frac{2}{ni}e^{in\cdot\Theta}\Big|_{-\theta_m/2}^{+\theta_m/2}=\quad\sum_{n=0}^{\infty}\frac{2}{ni}\left(e^{in\cdot\theta_m/2}-e^{-in\cdot\theta_m/2}\right) \qquad \text{(III.F)}$$

In Figure 26 we plot the left hand side of the GTE (III.F) and the first guess right hand side (III.G) on the same graph. For the latter we use the sum of the first 100 harmonics. We identify these as $LHS(\theta)$ and $RHS(\Theta)$ and they represent $\int f(\theta)d\theta$ and $\int F(\Theta)d\Theta$ respectively. Note that the left hand side represents the arc measurement of a circle of radius 1 in real space as shown in Figure 20. We are trying to find a function for the right hand side that is graphically equivalent to this.

$$LHS(\theta_m) := \theta_m \qquad RHS(\theta_m) := \sum_{n=1}^{100} \left[\frac{2}{n \cdot i} \cdot \left(e^{in\frac{\theta_m}{2}} \right) - \frac{2}{(n \cdot i)} \cdot \left(e^{-in\frac{\theta_m}{2}} \right) \right]$$

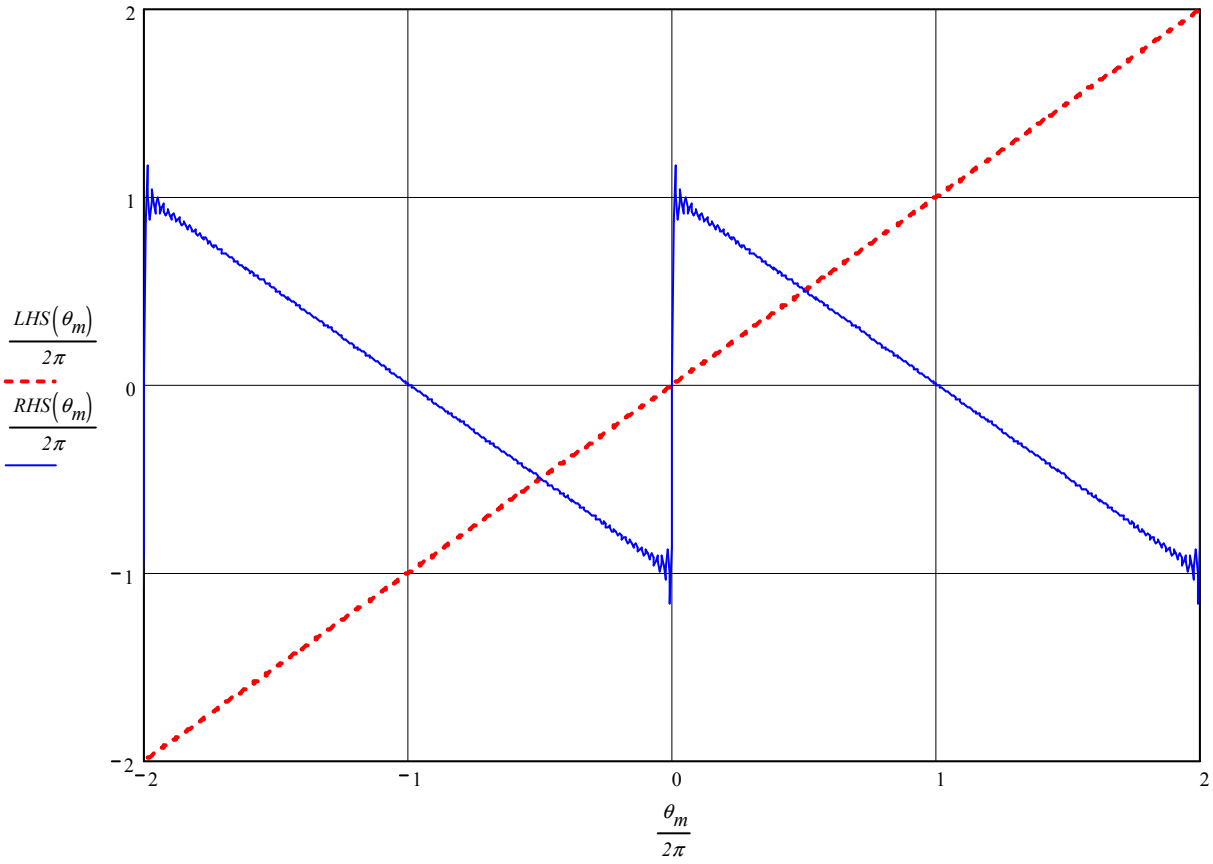


Figure 26 - The left hand side and first guess right hand side of the GTE are plotted. The left side, $\int f(\theta)d\theta$ is graphed using $f(\theta) = 1$ and appears as a straight line passing though the origin as per Figure 20. This represents arc measurement in real space. The right side, $\int F(\Theta)d\Theta$ is graphed using $F(\Theta) = \sum_{n=0}^{\infty} 2 \cdot e^{in \cdot \Theta}$ as a trial value. The first 100 harmonics were summed. It is seen that the right hand side approaches a straight line and the slope is correct but needs to be mirrored and shifted to be graphically equivalent with the left hand side.

We want the two functions to have the same value at all points in the interval. That means the two graphs need to be identical, or that $\int F(\Theta)d\Theta$ is graphically equivalent to $\int f(\theta)d\theta$.

It is seen from the figure that with only 100 harmonics $\int F(\Theta)d\Theta$ approaches a straight line. The slope is correct, but it needs to be mirrored and shifted to correctly model the real action.

Based on this result we now make a second guess. We will try $b=-1$ and $a=-2,$ and $c=\pi.$

Our second guess is: $f(\theta)=1$ and $F(\Theta)=\sum_{n=0}^{\infty}-2\cdot e^{in\cdot(-\Theta+\pi)}$ (III.G)

For the left side of the GTE

$$\int_{-\theta_m/2}^{+\theta_m/2} f(\theta)d\theta=\int_{-\theta_m/2}^{+\theta_m/2} 1\cdot d\theta=\theta\Big|_{-\theta_m/2}^{+\theta_m/2}=\theta_m \qquad \text{(III.H)}$$

For the right side of the GTE

$$\int_{-\theta_m/2}^{+\theta_m/2} \mathrm{F}(\Theta)d\Theta=\int_{-\theta_m/2}^{+\theta_m/2}\sum_{n=0}^{\infty}-2\cdot e^{in\cdot(-\Theta+\pi)}d\Theta=$$

$$\sum_{n=0}^{\infty}\frac{2}{in}e^{in\cdot(-\Theta+\pi)}\Big|_{-\theta_m/2}^{+\theta_m/2}=\sum_{n=0}^{\infty}\frac{2}{in}\left(e^{ni\cdot(\frac{-\theta_m}{2}+\pi)}-e^{ni\cdot(\frac{+\theta_m}{2}+\pi)}\right) \qquad \text{(III.i)}$$

In Figure 27 we again plot the left hand side $LHS(\theta)$, and the second trial right hand side first 100 harmonics $RHS(\Theta)$, which represent $f(\theta)$ and $F(\Theta)$ in (III.H) and (III.i).

We see from the figure that the $\int F(\Theta)d\Theta$ does converge to $\int f(\theta)d\theta$. Therefore $F(\Theta)$ is the correct function for mapping the real space function $f(\theta)$ in harmonic space. Both maps to give the same result over the entire range and are therefore graphically equivalent as $n\to\infty.$

$$LHS(\theta_m) := \theta_m \qquad RHS(\theta_m) := \sum_{n=1}^{100} \left[\frac{2}{n \cdot i} \cdot \left[e^{in \cdot \left(\frac{-\theta_m}{2} + \pi \right)} \right] - \frac{2}{n \cdot i} \cdot \left[e^{in \cdot \left(\frac{\theta_m}{2} + \pi \right)} \right] \right]$$

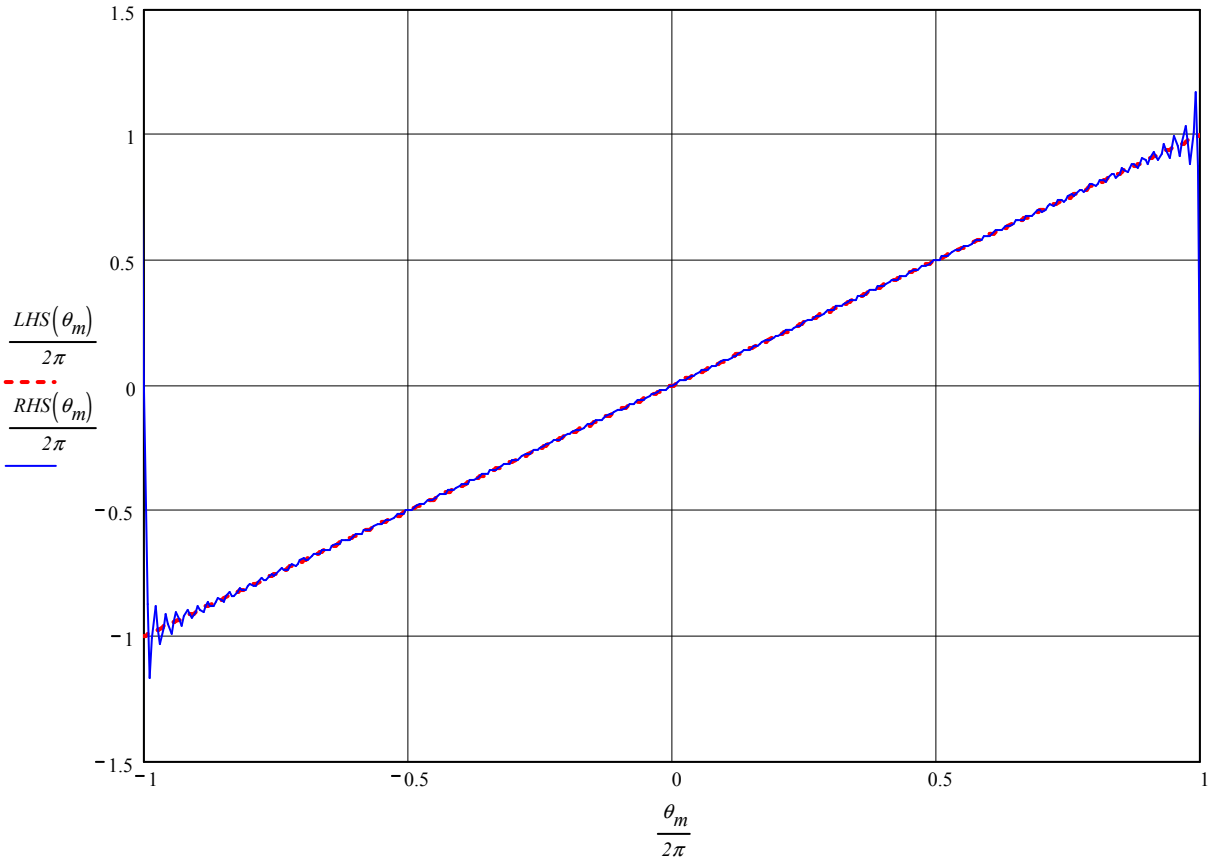


Figure 27 - $\int F(\Theta) d\Theta$ is graphed using the second trial value $F(\Theta) = \sum_{n=0}^{\infty} -2 \cdot e^{in \cdot (-\Theta + \pi)}$ summing the first 100 harmonics and appears to converge on $\int f(\theta) d\theta$ as $n \to \infty$. These functions are graphically equivalent.

In Figure 28 we demonstrate how as $n \to \infty$ the wave functions from the right side of the GTE sum to converge to the linear function (real measurement) of the left side of the GTE.

The graph has seven traces. The first trace is the left hand side of the GTE. The remaining six depict the following harmonics: 1, $1+2$, $\sum 1 \ldots 3$, $\sum 1 \ldots 10$, $\sum 1 \ldots 100$, and $\sum 1 \ldots 1,000$, the last being the thin line that is nearly straight. It closely models the left hand side. The harmonics approach the real function as $n \to \infty$.

$$IntF_1(\theta_m) := \frac{2}{1\cdot i}\cdot\left[e^{1\cdot i\cdot\left(\frac{-\theta_m}{2}+\pi\right)}\right] - \frac{2}{1\cdot i}\cdot\left[e^{1\cdot i\cdot\left(\frac{\theta_m}{2}+\pi\right)}\right]$$

$$SumF_2(\theta_m) := \sum_{n=1}^{2}\left[\frac{2}{n\cdot i}\cdot\left[e^{in\cdot\left(\frac{-\theta_m}{2}+\pi\right)}\right] - \frac{2}{n\cdot i}\cdot\left[e^{in\cdot\left(\frac{\theta_m}{2}+\pi\right)}\right]\right]$$

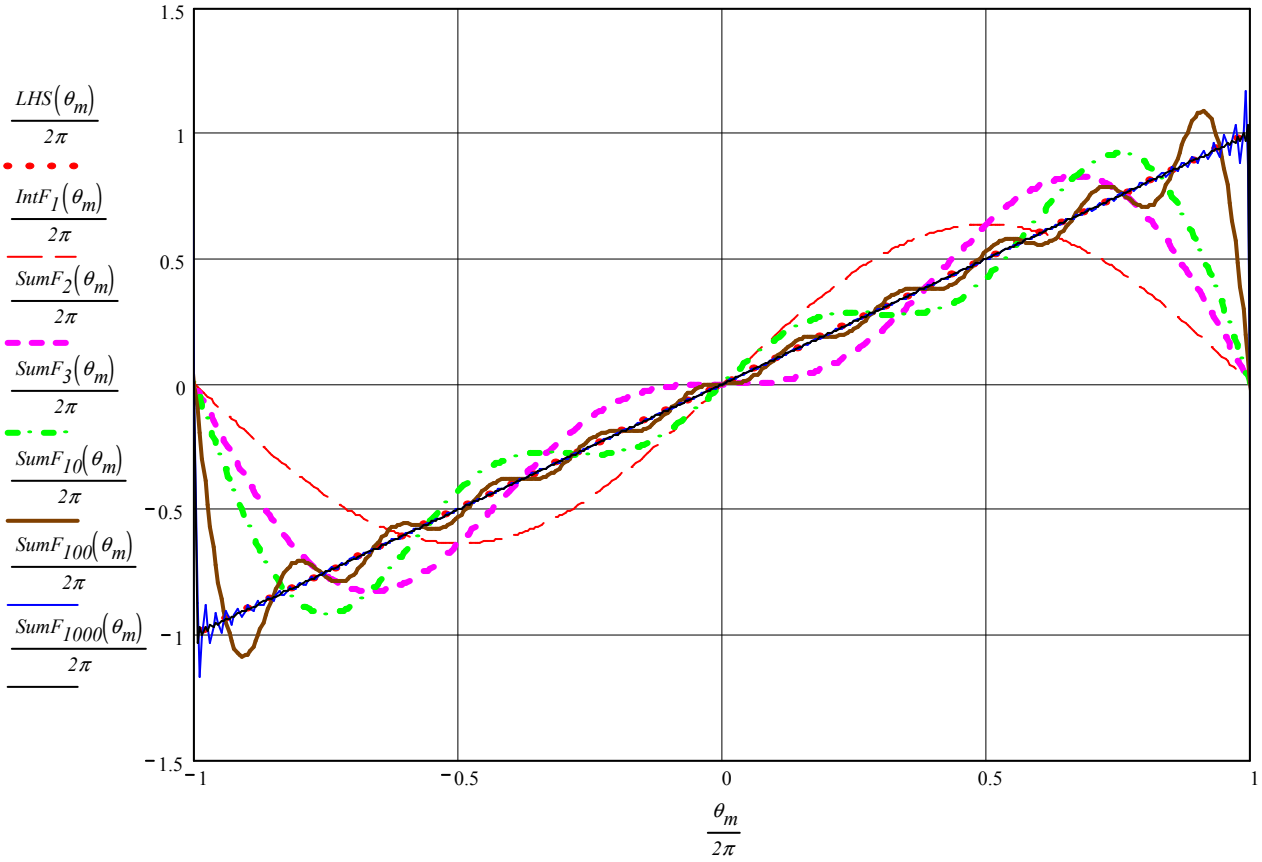


Figure 28 – Illustration of how the harmonics sum to produce an equivalent function to that found in nature. The left hand side of the GTE is shown harmonic 1 and sums of harmonics 1 to 2, 3, 10, 100, & 1,000. As more harmonics are included the sum of harmonics more closely matches the real measurement. Each subsequent harmonic decrease in amplitude as does its effect. As $n \to \infty$, $\int F(\Theta)d\Theta \to \int f(\theta)d\theta$. This indicates that $F(\Theta) = \sum_{n=0}^{\infty} -2\cdot e^{in\cdot(-\Theta+\pi)}$ satisfies the GTE.

Thus we have found the function $F(\Theta)$ that we needed to satisfy the equivalence relation.

Now we will verify that this value of $F(\Theta)$ does generate the Gregory Series when inserted into the GTE.

We again evaluate $\int f(\theta)d\theta$ at boundaries $-\pi/2$ to $+\pi/2$ to create a measured angle of $\theta_m = \pi$ as shown in Figure 25.

So for the left side of GTE :

$$\int_{-\pi/2}^{+\pi/2} f(\theta)d\theta = \int_{-\pi/2}^{+\pi/2} 1 \cdot d\theta = \theta\Big|_{-\pi/2}^{+\pi/2} = \pi \qquad \text{(III.J)}$$

For the right side, and using the identity $e^{i3\pi/2} = e^{-i\pi/2}$:

$$\int_{-\pi/2}^{+\pi/2} \mathrm{F}(\Theta)d\Theta = \int_{-\pi/2}^{+\pi/2} \sum_{n=0}^{\infty} -2 \cdot e^{in\cdot(-\Theta+\pi)}d\Theta =$$

$$\sum_{n=0}^{\infty} \frac{2}{in} e^{in\cdot(-\Theta+\pi)}\Big|_{-\pi/2}^{+\pi/2} = \sum_{n=0}^{\infty} \frac{2}{in}\left(e^{ni\cdot(\frac{-\pi}{2}+\pi)} - e^{ni\cdot(\frac{+\pi}{2}+\pi)} \right) =$$

$$\sum_{n=0}^{\infty} \frac{2}{in}\left(e^{ni\cdot(\pi/2)} - e^{ni\cdot(3\pi/2)}\right) = \sum_{n=0}^{\infty} \frac{2}{in}\left(e^{ni\cdot(\pi/2)} - e^{-ni\cdot(\pi/2)}\right) = \sum_{n=0}^{\infty} \frac{2}{in}\left(e^{ni\cdot(\pi/2)} - e^{-ni\cdot(\pi/2)}\right) \qquad \text{(III.K)}$$

Equating left (III.J) and right (III.K) sides of the GTE:

$$\pi = \sum_{n=0}^{\infty} \frac{2}{in} \cdot (e^{in\cdot(\pi/2)} - e^{-in\cdot(\pi/2)}) \qquad \text{(III.L)}$$

Since π is equal to four times the sum of the series, this implies each term of the series is given

$$f_n = \left(\frac{1}{4}\right) \cdot \left(\frac{2}{in} \cdot (e^{in\cdot(\pi/2)} - e^{-in\cdot(\pi/2)})\right) \qquad \text{(III.M)}$$

And in fact it does. Figure 29 shows how f_n generates the Gregory Series by subtracting the two functions.

We recognize the series in the right column of Figure 29 as $\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right)$. (III.N)

(To show the process the term F_n is calculated first, and then divided by 4 to determine f_n).

$$F(n) := \frac{2\cdot\left[e^{i\cdot n\cdot\left(\frac{\pi}{2}\right)}\right]}{i\cdot n} - \frac{2\cdot\left[e^{-i\cdot n\cdot\left(\frac{\pi}{2}\right)}\right]}{i\cdot n} \qquad \sum_{n=1}^{10000} F(n) = 3.141393$$

n =	A $\frac{2\cdot\left[e^{i\cdot n\cdot\left(\frac{\pi}{2}\right)}\right]}{i\cdot n}$ =	B $\frac{2\cdot\left[e^{-i\cdot n\cdot\left(\frac{\pi}{2}\right)}\right]}{i\cdot n}$ =	C F(n) =	C/4 f(n) = $\frac{F(n)}{4}$ =	*Coefficients*
1	2	-2	4	1	$= +1$
2	i	i	0	0	
3	-0.667	0.667	-1.333	-0.333	$= -\frac{1}{3}$
4	-0.5i	-0.5i	0	0	
5	0.4	-0.4	0.8	0.2	$= +\frac{1}{5}$
6	0.333i	0.333i	0	0	
7	-0.286	0.286	-0.571	-0.143	$= -\frac{1}{7}$
8	-0.25i	-0.25i	0	0	
9	0.222	-0.222	0.444	0.111	$= +\frac{1}{9}$
10	0.2i	0.2i	0	0	

Figure 29 – Calculation of (III.L) and (III.M) by tabulating each n value. Column A is the first term of Equation (III.L) . Column B is the second term. These are the two spinners we discuss later. Column C is the first minus the second as per (III.L). This is divided by four to determine the last tabulated column. We recognize these coefficients as the Gregory Series

$$\pi/4 = \left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots\right).$$

Also shown is the sum of the first 10,0000 harmonics. This approaches π as n approaches infinity.

Returning to (III.L) we now expand the summation to investigate how the mechanics operates:

$$\pi = \int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta = \sum_{n=0}^{\infty}\frac{2}{in}\cdot(e^{in\cdot(\pi/2)} - e^{-in\cdot(\pi/2)}) =$$

$$2\cdot\left[\left(\frac{1}{i}\cdot e^{1i\pi/2} - \frac{1}{i}\cdot e^{-1i\pi/2}\right) + \left(\frac{1}{2i}\cdot e^{2i\pi/2} - \frac{1}{2i}\cdot e^{-2i\pi/2}\right) + \left(\frac{1}{3i}\cdot e^{3i\pi/2} - \frac{1}{3i}\cdot e^{-3i\pi/2}\right) + \left(\frac{1}{4i}\cdot e^{4i\pi/2} - \frac{1}{4i}\cdot e^{-4i\pi/2}\right) + \ldots\right]$$

(III.O)

Recalling our definition of 2π as the angle at which the circle closes in the expression $e^{i\Theta}$, this implies that $e^{i2\pi} = +1$, and by extension $e^{i\pi} = -1$, $e^{i\pi/2} = +i$, and $e^{i3\pi/2} = -i$.

These are our "compass directions". This is simply based on our definition of 2π and does not assign a value to π. The $+1, +i, -1, -i$ pattern continues on a cycle of 2π so that $e^{i5\pi/2} = +i$, $e^{i3\pi} = -1$, $e^{i7\pi/2} = -i$, and $e^{i4\pi} = +1$.

Substituting these into the above we obtain $\pi = \int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta =$

$$2\cdot\left[\left(\frac{1}{i}\cdot(i)-\frac{1}{i}\cdot(-i)\right)+\left(\frac{1}{2i}\cdot(-1)-\frac{1}{2i}\cdot(-1)\right)+\left(\frac{1}{3i}\cdot(-i)-\frac{1}{3i}\cdot(+i)\right)+\left(\frac{1}{4i}\cdot(+1)-\frac{1}{4i}\cdot(+1)\right)+...\right] \tag{III.P}$$

which simplifies to $\pi = \int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta =$

$$2\cdot\left[\left(+1-(-1)\right)+\left(\frac{-1}{2i}-\left(\frac{-1}{2i}\right)\right)+\left(\frac{-1}{3}-\left(\frac{+1}{3}\right)\right)+\left(\frac{+1}{4i}-\left(\frac{1}{4i}\right)\right)+\left(\frac{+1}{5i}-\left(\frac{-1}{5i}\right)\right)+\left(\frac{-1}{6i}-\left(\frac{-1}{6i}\right)\right)+\left(\frac{-1}{7i}-\left(\frac{+1}{7i}\right)\right)+..\right]$$

$$=2\cdot\left[\left(+1-(-1)\right)+\left(\frac{-1}{2i}+\frac{+1}{2i}\right)+\left(\frac{-1}{3}-\frac{+1}{3}\right)+\left(\frac{+1}{4i}-\frac{+1}{4i}\right)+\left(\frac{+1}{5}-\frac{-1}{5}\right)+\left(\frac{-1}{6i}+\frac{+1}{6i}\right)+\left(\frac{-1}{7}-\frac{+1}{7}\right)+...\right] \tag{III.Q}$$

But $\left(\frac{-1}{2i}+\frac{+1}{2i}\right)=0,$ $\left(\frac{-1}{4i}+\frac{+1}{4i}\right)=0,$ $\left(\frac{-1}{6i}+\frac{+1}{6i}\right)=0,$ ....etcetera

We see that each pair of terms generated by an even value for *n* sums to zero and that when *n* is odd the components add, but also alternate in sign. Thus:

$$\pi = \int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta = 2\cdot\left[\left(+1-(-1)\right)+\left(\frac{-1}{3}-\frac{+1}{3}\right)+\left(\frac{+1}{5}-\frac{-1}{5}\right)+\left(\frac{-1}{7}-\frac{+1}{7}\right)+\left(\frac{+1}{9}-\frac{-1}{9}\right)...\right] \tag{III.R}$$

Or $$\pi = \int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta \;=\; 4\cdot\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-...\right) \tag{III.S}$$

Or $$\pi = \int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta \;=\; 4\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \tag{III.T}$$

And we have derived the Gregory Series for π as we set forth to do.

4. Modal Harmonic Equations for π

Let $f(\theta)$ be the function in real space for determining arc length along a circle of $r = 1$ with θ being the angle turned. Let $F(\Theta)$ be the corresponding function in harmonic space that we must determine. Let θ_m be the specific angle measured. Then the boundary values *A* and *B* are given by $-\frac{1}{2}\theta_m$ and $+\frac{1}{2}\theta_m$ respectively. There is an equivalence relation between real space and harmonic space maps of the action $A \rightarrow B$ called the General Transform Equation or GTE:

$$\text{GTE} \qquad \int_A^B f(\theta)d\theta = \int_A^B F(\Theta)d\Theta \qquad \text{(III.0)}$$

We are looking for the function $F(\Theta)$ which solves the GTE. The integral of that function in harmonic space must have the same value as the integral of $f(\theta)$ in real space. The general form of the right hand side is:

$$F(\Theta) = \sum_{n=0}^{\infty} a \cdot e^{in\cdot(b\cdot\Theta+c)} \qquad \text{(III.1)}$$

This is a set of harmonic wave functions where *n* is the harmonic wave number and a, b and c are normalization constants. We will use this in the right hand side of the GTE. The left hand side is the arc length measurement in real space. So $f(\theta) = 1$ since $r = 1$ and arc length equals $\int r \cdot d\theta$.

We set the limits of integration, or boundary values *A* and *B*, to $-\pi/2$ and $+\pi/2$. This creates an interval of measurement $\theta_m = \pi$ centered on zero. Then (III.0) can be written as

$$\int_{-\pi/2}^{+\pi/2} d\theta = \int_{-\pi/2}^{+\pi/2} \sum_{n=0}^{\infty} a \cdot e^{in\cdot(b\Theta+c)} d\Theta \qquad \text{(III.2)}$$

This relation defines a coding pattern for the harmonic components which allows them to combine in such a way that the two maps agree. n is the harmonic number and determines <u>*how much of each individual component is used*</u> in the sum. *b* is the frequency constant and is related to the boundary values (*A, B*). We will see that real and rational solutions exist for each *n* when

$$b \cdot \theta_m = \pm\pi \qquad \text{(III.3)}$$

Selecting these values in (III.2) results in all of the imaginary components from the right side of (III.2) to cancel out leaving only real numbers. Without the requirement of (III.3) the solutions could be complex and irrational as they could exist anywhere on the unit circle. The relation (III.3) restricts Θ to the quadrants of the circle and $e^{i\Theta}$ can only have the values $+1$ and -1. Thus solutions exist only at two compass points. This results in only integer ratios occurring, and the even numbered harmonics, which have imaginary values, all subtract out to zero. <u>*This determines which of the harmonic components are included and which are not used.*</u> These two constants together with n combine the individual wave components in the correct way to create the linear relationship of Figure 20.

a is the amplitude constant . It is a single factor that applies to the entire sum. It could double or half the value for example. So while *n* and b create the linear function, a determines the slope. The value of a is related to θ_m and b by the (III.2) with boundary values $A = -\frac{1}{2}\theta_m$ and $B = +\frac{1}{2}\theta_m$:

$$\theta_m = \frac{a}{b} \cdot (\pi / 2) \tag{III.4}$$

for equivalence in (III.2) so a must be either $+2$ or -2 in this case.

c is the phase factor. It offsets the result left or right but does not change the linear properties or slope.

We select $a = -2,\ b = -1,$ and $c = \pi$ and set the limits of integration, or boundary values *A* and *B*, to $-\pi / 2$ and $+\pi / 2.$ (See Section 6). This creates an interval of measurement $\theta_m = \pi$ centered on zero. Thus

$$f(\theta) = 1 \qquad \text{and} \qquad F(\Theta) = \sum_{n=0}^{\infty} -2 \cdot e^{in \cdot (-\Theta + \pi)} \tag{III.5}$$

Substituting into the GTE (III.0):

$$\int_{-\pi/2}^{+\pi/2} d\theta = \int_{-\pi/2}^{+\pi/2} \sum_{n=0}^{\infty} -2 \cdot e^{in \cdot (-\Theta + \pi)} d\Theta \tag{III.6}$$

So for the <u>left hand side</u> of the GTE we find:

$$\int_{-\pi/2}^{+\pi/2} f(\theta) d\theta = \int_{-\pi/2}^{+\pi/2} d\theta = \theta \Big|_{-\pi/2}^{+\pi/2} = \pi \tag{III.7}$$

Performing the integration and using $e^{+in \cdot (3\pi/2)} = e^{-in \cdot (\pi/2)}$ we find that for the <u>right hand side</u> of (III.1):

$$\int_{-\pi/2}^{+\pi/2} F(\Theta) d\Theta = \sum_{n=0}^{\infty} \frac{2}{in} \cdot e^{in \cdot (-\Theta + \pi)} \Big|_{-\pi/2}^{+\pi/2} = \sum_{n=0}^{\infty} \frac{2}{in} \cdot (e^{+in \cdot (\pi - \pi/2)} - e^{+in \cdot (\pi + \pi/2)}) =$$

$$\sum_{n=0}^{\infty} \frac{2}{in} \cdot (e^{+in \cdot (\pi/2)} - e^{+in \cdot (3\pi/2)}) = \sum_{n=0}^{\infty} \frac{2}{in} \cdot (e^{+in \cdot (\pi/2)} - e^{-in \cdot (\pi/2)}) \tag{III.8}$$

Substituting the values from (III.7) and (III.8) into (III.0):

$$\pi = \sum_{n=0}^{\infty} \frac{2}{in} \cdot \left(e^{+in\cdot(\pi/2)} - e^{-in\cdot(\pi/2)} \right) \tag{III.9}$$

In Figure 29 it is shown that this gives the correct value for π.

Now $e^{+in\cdot(\pi/2)} = \left(e^{i\cdot\pi/2}\right)^n$ and $e^{+in\cdot(3\pi/2)} = \left(e^{i\cdot\pi/2}\right)^{-n}$, so (III.9) can be rewritten as

$$\pi = 2 \cdot \sum_{n=0}^{\infty} \frac{1}{ni} \cdot \left[\left(e^{i\pi/2} \right)^{+n} - \left(e^{i\pi/2} \right)^{-n} \right] \tag{III.10A}$$

Recalling from our definitions that $e^{i\pi/2} = i$ this can be rewritten as

$$\pi = 2 \cdot \sum_{n=0}^{\infty} \frac{1}{ni} \cdot \left[(i)^{+n} - (i)^{-n} \right] \tag{III.10B}$$

But $\dfrac{(i)^n}{i} = (i)^{n-1}$ and $\dfrac{(i)^{-n}}{i} = (i)^{-n-1}$ so this can be simplified to

$$\pi = 2 \cdot \sum_{n=0}^{\infty} \frac{1}{n} \cdot \left[(i)^{n-1} - (i)^{-n-1} \right] \tag{III.10C}$$

For all even n that term cancels out. This includes $n = 0$. We will start the summation at $n = 1$. This avoids looking at the $n = 0$ term which has a $1/0$ factor. All even values cancel out including $n = 0$ so we can do this. Thus

$$\pi = 2 \cdot \sum_{n=1}^{\infty} \frac{1}{n} \cdot \left[(i)^{n-1} - (i)^{-n-1} \right] \tag{III.10D}$$

Expanding the series we obtain

$$\pi = 2 \cdot \left[1 \cdot \left((i)^0 - (i)^{-2} \right) + \frac{1}{2} \cdot \left((i)^1 - (i)^{-3} \right) + \frac{1}{3} \cdot \left((i)^2 - (i)^{-4} \right) + \frac{1}{4} \cdot \left((i)^3 - (i)^{-5} \right) + \frac{1}{5} \cdot \left((i)^4 - (i)^{-6} \right) + \ldots \right]$$

(III.10E)

Recall from Part II that $i^0 = +1$, as does i^{-8}, i^{-4}, i^{+4}, i^{+8} …. ; also that $i^1 = i$, as does i^{-7}, i^{-3}, i^{+5} …. ; $i^2 = -1$, as does i^{-6}, i^{-2}, i^{+6} ... ; and $i^3 = -i$ as does i^{-5}, i^{-1}, i^{+7} …. Therefore:

$$\pi = 2 \cdot \left[1 \cdot \left(1 - (-1) \right) + \frac{1}{2} \cdot (i - i) + \frac{1}{3} \cdot \left((-1) - (+1) \right) + \frac{1}{4} \cdot \left((-i) - (-i) \right) + \frac{1}{5} \cdot \left(1 - (-1) \right) + \ldots \right] \tag{III.10F}$$

This simplifies to

$$\pi = 2 \cdot \left[(1 \cdot 2) + \left(\frac{1}{2} \cdot 0\right) - \left(\frac{1}{3} \cdot 2\right) + \left(\frac{1}{4} \cdot 0\right) + \left(\frac{1}{5} \cdot 2\right) + \left(\frac{1}{6} \cdot 0\right) - \left(\frac{1}{7} \cdot 2\right) \ldots \right] \qquad \text{(III.10G)}$$

Or

$$\pi = 4 \cdot \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right) \qquad \text{(III.10H)}$$

This can be written as

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \qquad \text{(III.11)}$$

This is the familiar Gregory series for π.

Summary of Procedure

To better explain what we have done, first we found the harmonic space equation $F(\Theta)$ that when integrated over harmonic space $d\Theta$ produces a function that behaves identically to the real space (θ space) function $f(\theta)$ when integrated over angle $d\theta$ for the same boundary conditions $A \rightarrow B$. The harmonic space equation is, of course, composed of the myriad of possible harmonics each one at a smaller amplitude and higher frequency than the one before it. Because the amplitudes get smaller and smaller, as n approaches infinity the contribution of additional harmonics approaches zero. Thus the infinite sum of waves $\int F(\Theta) d\Theta$ converges to the physical measurement $\int f(\theta) d\theta$. In practice we need only calculate only to the accuracy required.

When we establish the limits of integration $-\pi / 2$ and $+\pi / 2$ we are setting the boundary conditions of a differential equation that is written in integral form. This equation has eigenvalues which determine the pattern that needs to be applied to the harmonics to solve the differential equation. Like a vibrating string fixed at each end, the boundary conditions set rules on the permissible modes. In our case all of the terms where n is even cancel out, while the odd terms have a factor of $2 / n$ and alternate in sign.

Although we cannot view the harmonic space, the setting of boundary conditions establishes eigenvalues for the differential equation. It is a requirement of physics that mapping in the two systems yield the same result. The sum of the eigenvalues from harmonic space appears as a remnant in real space as π. The great paradox is that we cannot view harmonic space yet it determines the value of our fundamental constants.

In summary the arc length around a circle with $r=1$ is given by $\int_{-\pi/2}^{+\pi/2} f(\theta)d\theta = \pi$ and must equal $\int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta$ for the two mapping spaces to have the same result for a common event, and since

$$\int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta = -4\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \tag{III.12}$$

$$\text{then } \pi \quad \underline{\text{must equal}} \quad 4\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \tag{III.13}$$

Summary of Normalized Equations

With the normalization factors $a=-2,\ b=-1$, and $c=\pi$ determined, and $\theta_m=\pi$, $A=-\pi/2$ and $B=+\pi/2$ now established, we restate:

Returning to the GTE $$\int_A^B f(\theta)d\theta = \int_A^B F(\Theta)d\Theta \tag{III.14}$$

The left and right sides are of the form

$$f(\theta)=1 \qquad \text{and} \qquad F(\Theta)=\sum_{n=0}^{\infty} a\cdot e^{in\cdot(b\Theta+c)} \tag{III.14A}$$

Substituting $$\int_A^B f(\theta)d\theta = \int_A^B \sum_{n=0}^{\infty} a\cdot e^{in\cdot(b\Theta+c)}d\Theta \tag{III.15}$$

And with $$f(\theta)=1 \qquad \text{and} \qquad F(\Theta)=\sum_{n=0}^{\infty} -2\cdot e^{in\cdot(-\Theta+\pi)} \tag{III.16}$$

Substitution yields $$\int_{-\pi/2}^{+\pi/2} d\theta = \int_{-\pi/2}^{+\pi/2}\sum_{n=0}^{\infty} -2\cdot e^{in\cdot(-\Theta+\pi)}d\Theta \tag{III.17}$$

For the left hand side of the GTE:

$$\int_{-\pi/2}^{+\pi/2} f(\theta)d\theta = \int_{-\pi/2}^{+\pi/2} 1\cdot d\theta = \theta\Big|_{-\pi}^{+\pi} = \pi \tag{III.18}$$

For the right hand side we obtain:

$$\int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta = \sum_{n=0}^{\infty} \frac{2}{in} \cdot e^{in\cdot(-\Theta+\pi)}\Big|_{-\pi}^{+\pi} = \sum_{n=0}^{\infty} \frac{2}{in} \cdot (e^{+in\cdot(\pi/2)} - e^{-in\cdot(\pi/2)}) \qquad \text{(III.19)}$$

The expansion of this was shown in Equations (III.O) through (III.R) and is condensed here:

$$\int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta = \sum_{n=0}^{\infty} \frac{2}{in} \cdot (e^{in\cdot(\pi/2)} - e^{-in\cdot(\pi/2)}) =$$

$$2 \cdot \left[\left(\frac{1}{1i} \cdot e^{1i\pi/2} - \frac{1}{1i} \cdot e^{-1i\pi/2} \right) + \left(\frac{1}{2i} \cdot e^{2i\pi/2} - \frac{1}{2i} \cdot e^{-2i\pi/2} \right) + \left(\frac{1}{3i} \cdot e^{3i\pi/2} - \frac{1}{3i} \cdot e^{-3i\pi/2} \right) + \left(\frac{1}{4i} \cdot e^{4i\pi/2} - \frac{1}{4i} \cdot e^{-4i\pi/2} \right) + .. \right]$$

$$= 2 \cdot \left[\left(+1 - (-1) \right) + \left(\frac{-1}{3} - \frac{+1}{3} \right) + \left(\frac{+1}{5} - \frac{-1}{5} \right) + \left(\frac{-1}{7} - \frac{+1}{7} \right) + \left(\frac{+1}{9} - \frac{-1}{9} \right) ... \right]$$

$$= 4 \cdot \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - ... \right) = 4 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \qquad \text{(III.20)}$$

So for the left side of the GTE $$\int_{-\pi/2}^{+\pi/2} F(\Theta)d\Theta = 4 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \qquad \text{(III.21)}$$

Substituting left (III.18) and right (III.21) sides into the GTE (III.14):

$$\pi = 4 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \qquad \text{(III.22)}$$

This can be rearranged to $$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \qquad \text{(III.23)}$$

and we have the Gregory Series for $\pi / 4$.

5. Mechanics

How does the math work? Is there a mechanism that models all of this? The upper and lower boundaries $+\pi/2$ and $-\pi/2$ of the integral along with the constant $b=-1$ result in the factor $(e^{i\cdot\pi/2})^n-(e^{i\cdot\pi/2})^{-n}$ in Equation (III.10A). These two terms which are subtracted can be thought of as two separate spinners that are synchronized to move a quarter turn with each n, but they are moving in opposite directions. See Figure 30.

At the first harmonic $n=1$ the two spinners point towards each other, one with a value of $+i$ and the other $-i$. So $(+i)-(-i)=2i$. The factor $(e^{i\cdot\pi/2})^n-(e^{i\cdot\pi/2})^{-n}$ requires that we subtract the results from the two spinners.

For $n=2$ they point both point down, each with a value of -1 So $(-1)-(-1)=0$.
For $n=3$ they point away from each other, one with a value of $-i$ and the other $+i$. So $(-i)-(+i)=-2i$.
For $n=4$ they point both point up, each with a value of +1 So $(+1)-(+1)=0$.
At $n=5$ the pattern repeats to that of $n=1$.

We are subtracting the result of one spinner from the other. Therefore when both spinners point in the same direction those terms will cancel. This occurs for all of the even numbered harmonics as both spinners are in the same position, either up or down.

For the odd numbered harmonics the values of the two spinners are always of opposite sign. These are subtracted and therefore result in a non-zero value. At $n=1$ we subtract a negative value from a positive value and the result is positive. But at $n=3$ we subtract a positive value from a negative value and the result is negative. Therefore the odd numbered harmonics will alternate in sign.

The spinners track and determine which modes add, which subtract, and which cancel out. All the even numbered harmonics cancel out and the odd numbers double and alternate in sign. The amplitude decreases for each harmonic by a factor of $1/n$.

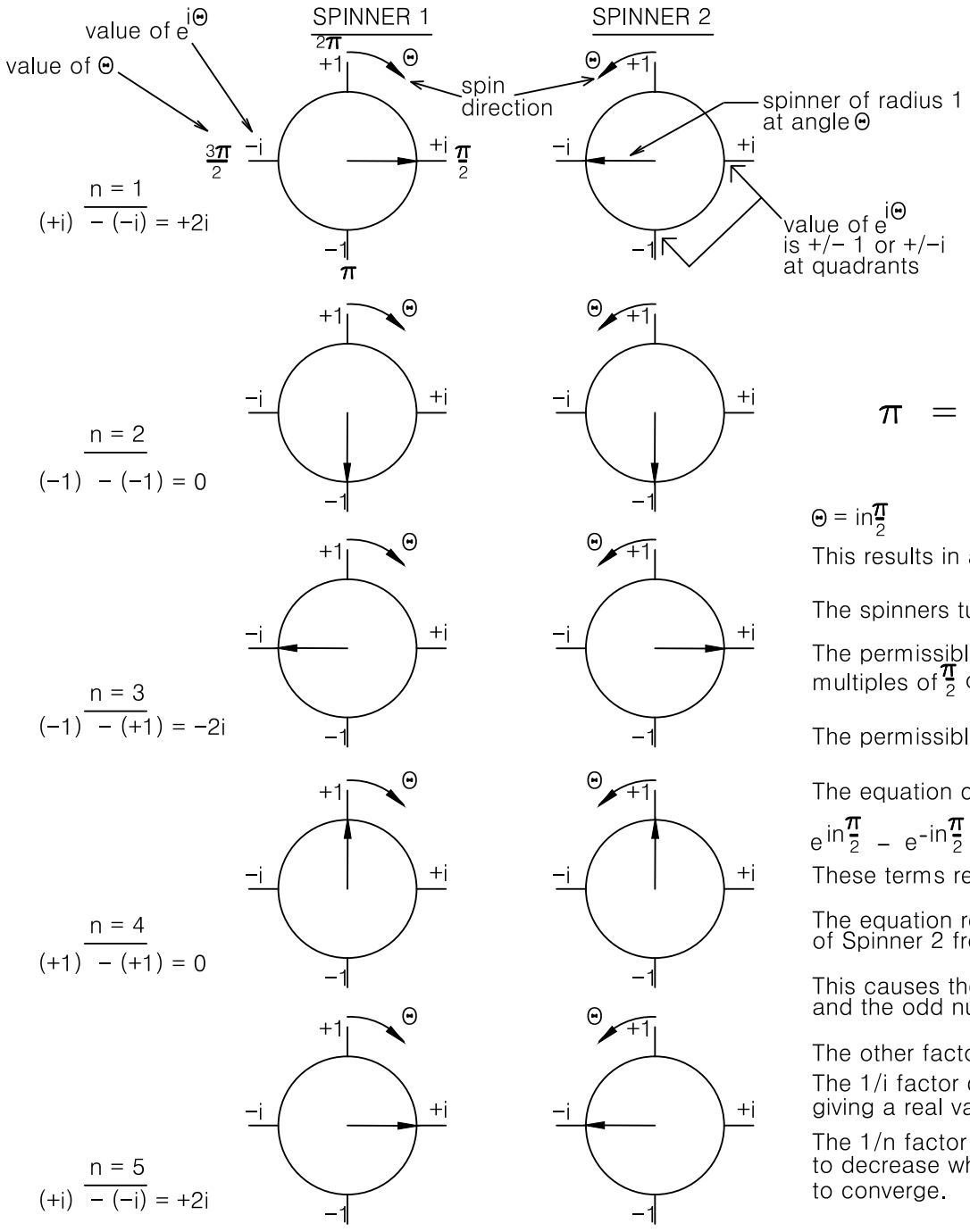
We are solving a differential equation that appears in integral form. It defines which harmonics to include and how much of each one in order to have the results in harmonic space agree with those of nature. There is a complete range of harmonics and all amplitudes are permissible, yet there is only a single solution.

The exact recipe for the amount of each mode that needs to be combined is encoded in the differential equation. It is a code that when applied to the harmonic spectrum results in the value of π. This is nature's means for storing that value and accessing it.

π has to have some value and there are insufficient dimensions in real space to provide this. Nature chose this method to establish that value. Whenever an action occurs involving the value of π that action is following natural law and obeying a differential equation in the same way that a falling body would follow its natural laws.

But π it is more subtle than G. π is dimensionless so there are no indications of what to associate it to. Our assumption is that it is the set of waves that is needed to create it. We have a model that is relatively easy to visualize with two opposing spinners that operate in virtual space.

CODING PATTERN FROM TWO OPPOSING SPINNERS



SPINNER 1 SPINNER 2

$$\pi = \sum_{n=1}^{\infty} \frac{2}{in}\left(\left(e^{i\frac{\pi}{2}}\right)^{n} - \left(e^{i\frac{\pi}{2}}\right)^{-n}\right)$$

$\Theta = in\frac{\pi}{2}$

This results in a quarter turn for each n

The spinners turn in opposite directions.

The permissible positions for the spinner are multiples of $\frac{\pi}{2}$ or vertical and horizontal only.

The permissible values for $e^{i\Theta}$ are +1, +i, −1, −1.

The equation contains the factor

$e^{in\frac{\pi}{2}} - e^{-in\frac{\pi}{2}}$ or equally $\left(e^{i\frac{\pi}{2}}\right)^{n} - \left(e^{i\frac{\pi}{2}}\right)^{-n}$

These terms represent the two opposing spinners.

The equation requires we subtract the result of Spinner 2 from Spinner 1.

This causes the even numbers to cancel out and the odd numbers to add but alternate in sign.

The other factor in the equation is 1/ni.

The 1/i factor cancels with i from the odd harmonics giving a real value.

The 1/n factor causes the contribution of each mode to decrease which is neccessary for the infinite sum to converge.

FIGURE 30

6. More on Normalization

The combination of normalization factors a, b, and c and measured angle θ_m which were used are not the only ones possible for generating this series. Other combinations will also generate the series.

θ_m determines the boundary values of the integral, $A = -\theta_m / 2$ and $B = +\theta_m / 2$. When $\theta_m = \pi$ and $b = -1$ this results in $\int_A^B \sum_{n=0}^{\infty} a \cdot e^{in\cdot(\mathrm{b}\Theta + c)} d\Theta$ producing the factor $e^{in\cdot(\pi/2)} - e^{-in\cdot(\pi/2)}$ which generates the series. This restricts the evaluation to spinner positions up and down only. Here the imaginary terms cancel out leaving only real numbers, and the ± 1 spinner value further restricts the evaluation to rational numbers only.

It was previously indicated that the normalization factors for this series follow the relation $b \cdot \theta_m = \pm\pi$. Other combinations of θ_m and b will also generate the series or a multiple of it. The amplitude and phase factors would be adjusted accordingly to result in graphical equivalence.

7. Generation of Other Series

Is it possible to use other patterns where the spinners do not land at the quadrants?

If this were to occur we would get irrational values for $e^{in\cdot(\Theta)}$ rather than $+1, -1, +i, -i$ only. We will try other values for b. In Figure 31 we compare $b = 1/2$ to $b = 1/3$ and $b = 1/4$.

We see that for $b = 1/2$: $\sum\int F_n = \pi$; For $b = 1/3$: $\sum\int F_n = 2\pi$; and for $b = 1/4$: $\sum\int F_n = 3\pi$. The value is one less than the reciprocal, times π. In general, $\sum\int F_n = (\mathrm{b}^{-1} - 1)\pi$.

Also we see that using a different value for b can generate a different series for π. For example $b = 1/3$ will generate the series:

$$\frac{2\pi}{3\sqrt{3}} = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - .... = \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{3n+1} + \frac{(-1)^n}{3n+2}\right] \qquad \text{(III.Z)}$$

It is interesting to note that in this series every third term equals zero. There are two terms of one sign, then zero, then two terms of the opposite sign. This occurs because the spinners land at intervals with value $\pm\sqrt{3}/2$, so the first two are on the right half of the circle (+ value), the third is down (0), and the next two are on the left side (- value). See Figure 32.

b=1/2

1. The integral on the right hand side of the GTE is:

$$IntF_n := \frac{2\left\lfloor e^{i\cdot n\cdot\left(\frac{\pi}{2}\right)}\right\rfloor}{i\cdot n} - \frac{2\left\lfloor e^{-i\cdot n\cdot\left(\frac{\pi}{2}\right)}\right\rfloor}{i\cdot n}$$

2. The value of this integral is:

$$\sum_{n=1}^{1000}\left[\frac{2\left\lfloor e^{i\cdot n\cdot\left(\frac{\pi}{2}\right)}\right\rfloor}{i\cdot n} - \frac{2\left\lfloor e^{-i\cdot n\cdot\left(\frac{\pi}{2}\right)}\right\rfloor}{i\cdot n}\right] = 3.14 \qquad = 1\pi$$

3. The equation also provides the combination of harmonic components are for each *n*:

$$\frac{IntF_n}{4} =$$

1
0
-0.333
0
0.2
0
-0.143
0
0.111
0

4. This combination can be written as a series:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \ldots.$$

5. This can be expressed in summation form:

$$\frac{\pi}{4} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

b=1/3

$$InfF_n := \frac{3\left\lfloor e^{i\cdot n\cdot\left(\frac{\pi}{3}\right)}\right\rfloor}{i\cdot n} - \frac{3\left\lfloor e^{-i\cdot n\cdot\left(\frac{\pi}{3}\right)}\right\rfloor}{i\cdot n}$$

$$\sum_{n=1}^{1000}\left[\frac{3\left\lfloor e^{i\cdot n\cdot\left(\frac{\pi}{3}\right)}\right\rfloor}{i\cdot n} - \frac{3\left\lfloor e^{-i\cdot n\cdot\left(\frac{\pi}{3}\right)}\right\rfloor}{i\cdot n}\right] = 6.283 \qquad = 2\pi$$

$$\frac{InfF_n}{3\sqrt{3}} =$$

1
0.5
0
-0.25
-0.2
0
0.143
0.125
0
-0.1

$$\frac{2\pi}{3\sqrt{3}} = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \ldots.$$

$$\frac{2\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{3n+1} + \frac{(-1)^n}{3n+2}\right]$$

b=1/4

$$IntF_n := \frac{4\left\lfloor e^{i\cdot n\cdot\left(\frac{\pi}{4}\right)}\right\rfloor}{i\cdot n} - \frac{4\left\lfloor e^{-i\cdot n\cdot\left(\frac{\pi}{4}\right)}\right\rfloor}{i\cdot n}$$

$$\sum_{n=1}^{1000}\left[\frac{4\left\lfloor e^{i\cdot n\cdot\left(\frac{\pi}{4}\right)}\right\rfloor}{i\cdot n} - \frac{4\left\lfloor e^{-i\cdot n\cdot\left(\frac{\pi}{4}\right)}\right\rfloor}{i\cdot n}\right] = 9.415 \qquad = 3\pi$$

$$\frac{IntF_n}{4\sqrt{2}} =$$

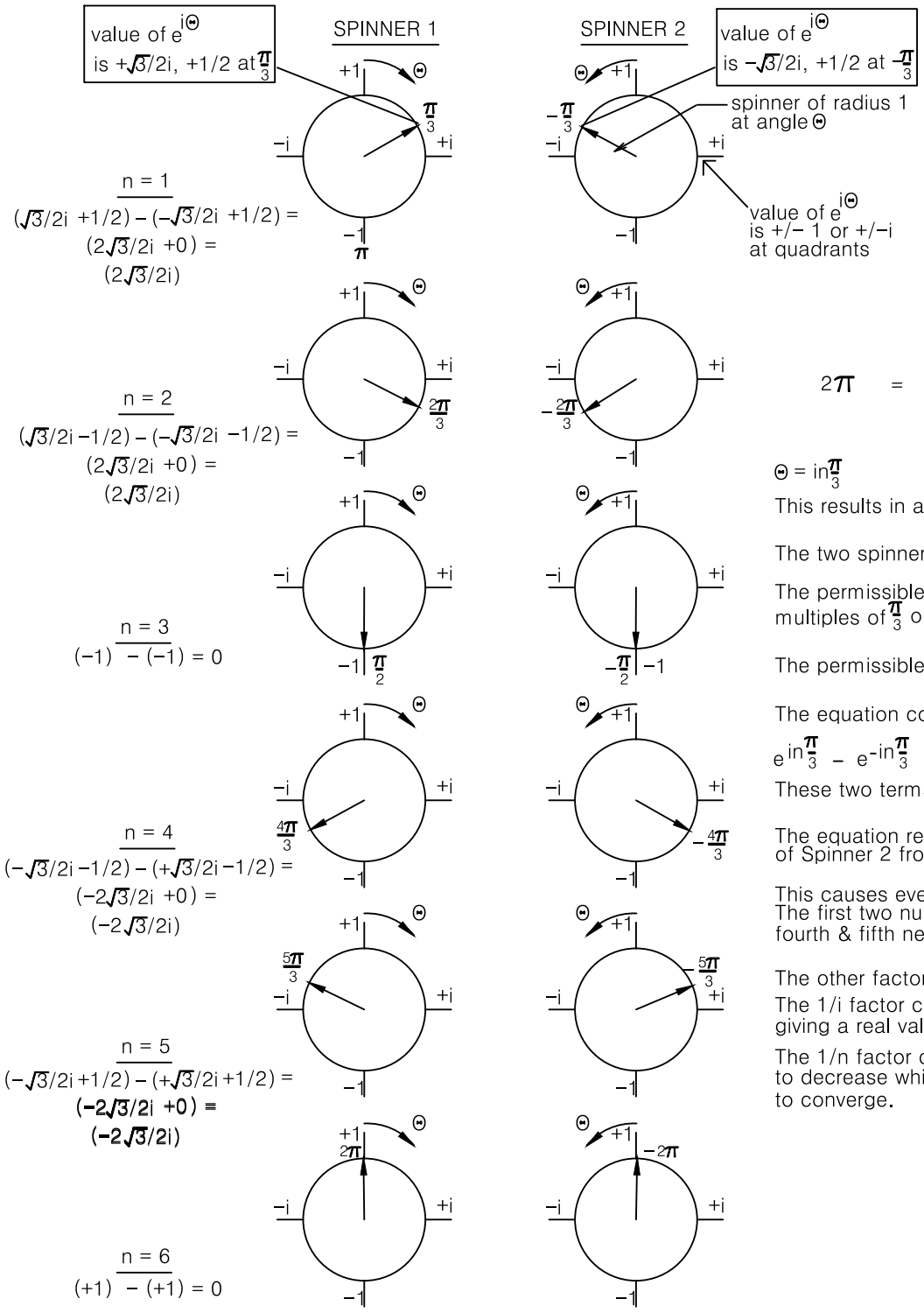
1
0.707
0.333
0
-0.2
-0.236
-0.143
0
0.111
0.141

$$\frac{3\pi}{4\sqrt{2}} = 1 + \frac{\sqrt{2}}{2} + \frac{1}{3} - \frac{1}{5} - \frac{\sqrt{2}}{6} - \frac{1}{7} + \ldots.$$

$$\frac{3\pi}{4\sqrt{2}} = \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{4n+1} + \frac{\sqrt{2}\cdot(-1)^n}{4n+2} + \frac{(-1)^n}{4n+3}\right]$$

Figure 31 - $b = 1/2,\ 1/3$ and $1/4$ in equation (III.W) result in different series for π.

CODING PATTERN FROM TWO OPPOSING SPINNERS



SPINNER 1 SPINNER 2

$$2\pi = \sum_{n=1}^{\infty} \frac{3}{in}\left(\left(e^{i\frac{\pi}{3}}\right)^{n} - \left(e^{i\frac{\pi}{3}}\right)^{-n}\right)$$

$\Theta = in\frac{\pi}{3}$

This results in a one sixth turn for each n.

The two spinners turn in opposite directions.

The permissible positions for the spinner are multiples of $\frac{\pi}{3}$ or 2'oclock, 4, 6, 8, 10, & 12 only.

The permissible values for $e^{i\Theta}$ are +1, +i, −1, −1.

The equation contains the factor

$e^{in\frac{\pi}{3}} - e^{-in\frac{\pi}{3}}$ or equally $\left(e^{i\frac{\pi}{3}}\right)^{n} - \left(e^{i\frac{\pi}{3}}\right)^{-n}$

These two terms represent the two spinners.

The equation requires we subtract the result of Spinner 2 from Spinner 1.

This causes every third number to cancel out. The first two numbers are positive, third zero, fourth & fifth negative, sixth zero...

The other factor in the equation is 1/ni.

The 1/i factor cancels with i from the non zero harmonics giving a real value.

The 1/n factor causes the contribution of each mode to decrease which is neccessary for the infinite sum to converge.

FIGURE 32