Part II. A. Debonking the Euler Equation $e^{i\pi} + 1 = 0$

In this section we debonk the dreaded Euler Equation. See Reference (1). It has driven many bonkers.

My brother Barry is a balloonist. In fact, he is the best balloonist in the world. He was born in a balloon and has spent his whole life there.

One day he called me on his cell phone. "There is something very peculiar going on here," he said. "I am watching a ball floating down the river. It goes fast but then slows down and stops, then speeds up again. The pattern keeps repeating."

I went down to the river to experience this strange phenomenon and this is what I saw. The river was wavy, and when the ball was at the top or bottom of the wave it was moving purely in a horizontal direction, but between these points some of the motion was up or down. And midway between top and bottom there was no horizontal motion at all, only vertical, making it appear motionless from the balloon.

This is illustrated in Figure 10. At positions 1, 4 and 6 the motion is completely horizontal: downstream. At position 3 the ball is moving downwards but not forwards. At position 5 it is moving upwards but not forwards. Position 2 has components in both the vertical and horizontal directions.

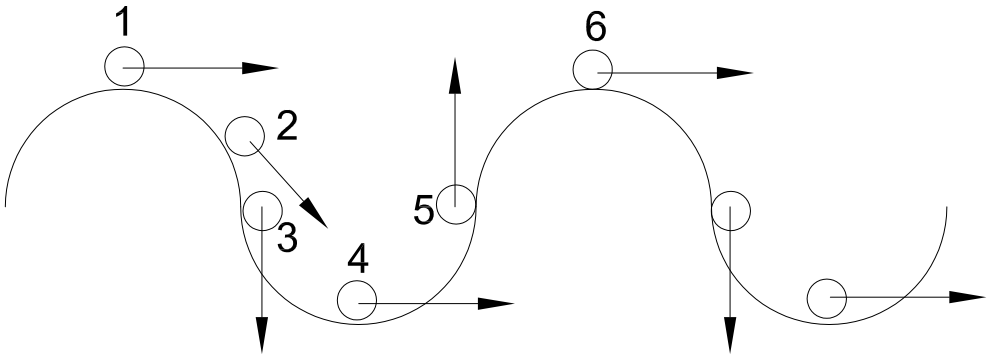

Figure 10 - Ball in Wave

From his balloon Barry could not see the waves or the vertical motion. He could only see the horizontal component of the velocity which cycled between a maximum and zero. I told him he needs to imagine the waves and the up and down motion they create. He was OK with that but wanted a mathematical way to describe it. This is what we came up with.

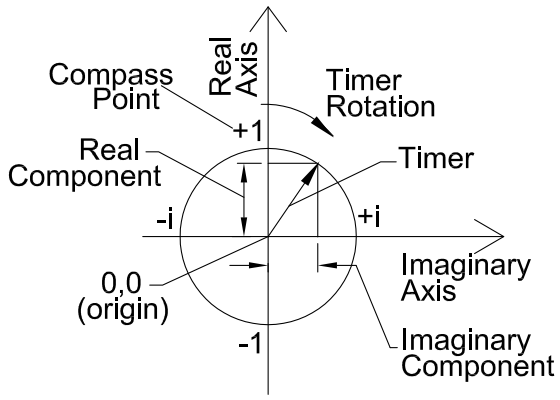

Figure 11A - The timer position indicates the real and imaginary components

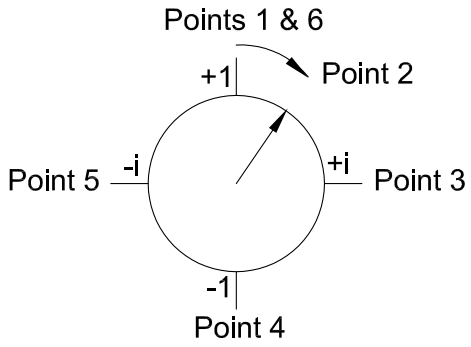

Figure 11B - Timer Position to wave Position in Figure 10

There was a dimension that Barry could not see, and the motion was dividing itself between the part he could see and the part he couldn't see. We set up a timer that spins in sync with the wave as shown in Figure 11. When it pointed perfectly up or down, all of the motion would be visible (horizontal). When it pointed exactly right or left, all of the motion would be in a direction he had to imagine (vertical), which we designated $+i$ and $-i$ on our timer. And between these the motion would be distributed amongst the two directions. Our timer, a one dimensional device, could tell us about motion in two dimensions. It is able to do this because of constraints. There is only so much water and it has to divide itself between vertical and horizontal motion. The vertical and horizontal motions are not independent, if we know one we can calculate the other.

The $+i$ and $-i$ positions indicate that all of the motion is in the imaginary direction, the one he could not see, and the +1 and -1 positions indicate that all of the motion is in the visible (real) direction. In between these four spots the velocity has both an imaginary and a real component.

The Compass Model

Thus far we have treated $+i$, $-i$ and +1,-1 as compass points. We have not assigned numerical values to them, they simply occur at the quarter points of the circle. These quarter points could be determined with simple geometric tools like a compass. In this type of model the closed circle is the base unit. We talk about half circles and quarter circles or cycles which are in fact multiples or fractions of the circle. When we use MHA to determine the value of π we will use this system. However for the moment we will diverge and look at the numerical properties of i the imaginary number that are used in math.

Imaginary and Complex Numbers

i is defined mathematically as the square root of -1. This is called an imaginary number and is frequently used to describe a second property that may be hidden. A complex number has a real part, A, and an imaginary part, B, and is of the form $A + Bi$. Nature uses these to control two different values with just one complex number.

In our example when the spinner points to +1 the real component A equals 1 and the imaginary component B equals zero. *Note that we always work with a circle of radius 1.* As the spinner rotates the real component decreases and the imaginary one increases. At $+i$ the imaginary component has a value of 1 and the real part is zero. The real and imaginary components represent the horizontal and vertical components of the velocity of the ball in the waves. Thus the operation of a single spinner controls two different properties that are related but have different values.

Raising to an Imaginary Power and the Formation of Spinners

When we raise any number to an imaginary power we create a spinner (or timer) of radius 1. This means that when evaluated it will yield a complex number $A + Bi$ which always has a magnitude of 1. As the pointer turns the values of A and B will change, but they will always have a resultant of 1. When it points up the value will be $1 + 0i$ which equals +1. When it points to the right the value will be $0 + 1i$ which equals $+1i$. When the spinner is somewhere between two compass points there is both a horizontal component and a vertical component. Starting at any value, after one cycle the spinner will return to that value. We show this in Figure 12. Here we raise some number x to the $i\theta$ power.

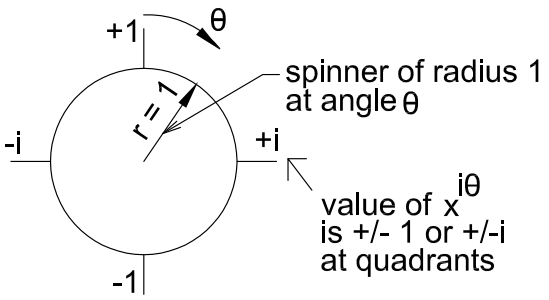

Figure 12A - Raising any number to the iθ power creates a spinner of radius 1. At the four quadrants there is only one component: +1, +i, -1, or -i.

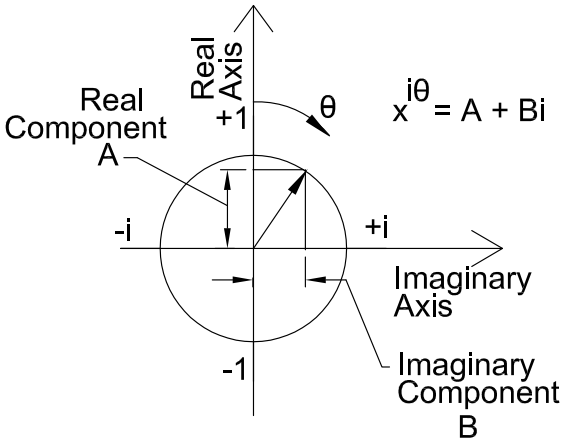

Figure 12B - When the spinner is between the two compass points there is a real and imaginary component

This spinner described by $x^{j\theta}$ models our timer in terms of resulting value. It describes the real and imaginary component velocities as cyclical and out of phase by half a cycle, meaning one is a maximum when the other is at minimum. However it is not calibrated in frequency. It is not in sync with the wave. How do we fix this?

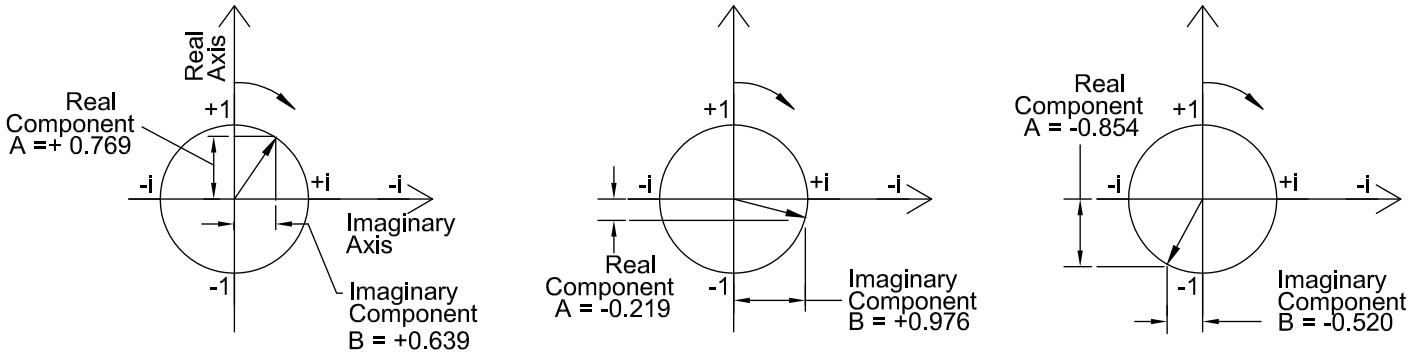

Figure 13 - The higher the base number the faster the spinner turns, but the magnitude is always 1.

Depending on our choice for the base number x, the spinner spins faster or slower, but the magnitude is always 1. In Figure 13 we show spinners at 2^i, 6^i, and 40^i to illustrate how higher base translates to a faster spinner.

What is the right number to use for the base? We will look at $e^{j\theta}$. That is because e^x is that special function that equals its own derivative. It is naturally occurring and is frequently used in the solution of differential equations.

So now we ask, at what angle θ will $e^{j\theta}$ close perfectly? That is, if we start at $\theta_0 = 0$, which is the +1 or North position, at what angle θ does it need to spin to return to +1 (North) again, to close and form a perfect circle?

That special angle is 2π. Every time the spinner turns an angle of 2π it returns to the starting position. **Thus 2π is the angle θ turned that causes $e^{i\theta}$ to close.**

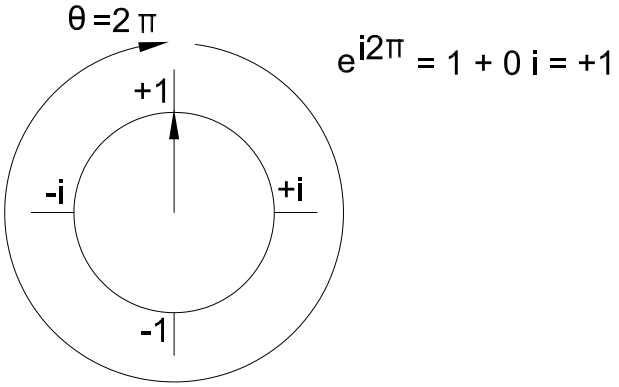

Figure 14 - 2 p is the angle θ that causes $e^{i\theta}$ to close.

Note that we always use an action such as a turn. We did this in our formulation of e but more subtly through the boundary conditions. One reason the fundamental constants have proven so difficult is that they have been studied as static.

π – The Ratio of Lengths or an Angle?

π is usually defined as the ratio of two lengths, the circumference to the diameter of a circle. **Instead we will define π in terms of the angle which we need to turn to achieve closure. Thus we define 2π as the angle θ turned that causes $e^{i\theta}$ to close.**

We consider the circle and more specifically its arc length at closure as a base unit. 2π is a full turn. So π is the angle one must turn to go half way around. This definition of π is fundamentally different from the traditional one. The ratio of circumference to diameter still holds true, and that conversion still applies since r = 1. However instead of working with two quantities (circumference and diameter) we only use one (angle turned).

The Euler Equation : $e^{i\pi}+1=0$

2π is the angle required to make a full turn around a circle, so: 1) π is the angle one must turn to go half way around , i.e. North to South. If we rearrange the Euler equation written above, by subtracting 1 from each side of the equation we get : 2) $e^{i\pi}=-1$. These two underlined statements 1) and 2) are identical in meaning.

We know that $e^{i\pi}$ represents a rotation of an angle π or half way around. What is the significance of this being equal to -1? If we start at North (+1) we will end up at South (-1). If we start at East (+*i*) we will end up at West (-*i*). In general, an arrow from the center to any point on the circle given by A + B*i*, would by be subject to a factor of -1 under this rotation. This causes the arrow to point in the opposite direction. This arrow is of length 1 since it goes from center to some point on the radius and r = 1. It is called a unit vector. Then A + B*i* $\Rightarrow$ - A - B*i* .

Thus the Euler Equation is a description of turning halfway around a circle and states that this will result in the application of a factor of -1 to the original unit vector. If we start at NNE we end up at SSW. The Euler equation is anything but mysterious – it only states the obvious. *If you turn around you will be looking in the opposite direction*. See Figure 15.

By extension a full turn or 2π would result in a factor of +1. In other words we would end up where we started from. This is the same as our definition that 2π is the angle turned that causes closure. The Euler Equation implies our definition and vica versa.

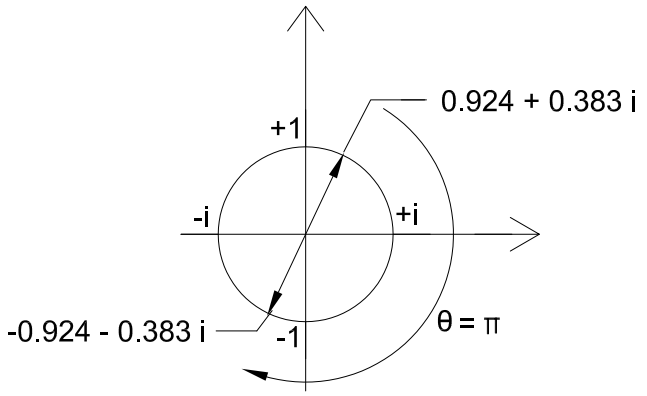

Figure 15 - The Euler Equation is another way of saying that if you turn around you will be looking in the other direction.

More on Compass Points

It is logical to refer to $e^{i\theta}$ at the quarter points of the circle as compass points as this reflects the underlying property of the circle as a base unit. The compass points we have been using are a natural consequence of our definition for 2π as the angle turned for closure. At +1 or the North marker, $\theta = 0$ and $e^{i\theta} = +1$. That is why we identified this point as +1. At +i or the East Marker $\theta = \pi/2$ and $e^{i\theta} = +i$. At -1 or the South Marker $\theta = \pi$ and $e^{i\theta} = -1$. At -i or the East Marker $\theta = 3\pi/2$ and $e^{i\theta} = -i$.

These quadrants of the circle line up exactly on the Cartesian axes so that one of the components is always zero and the other is always 1. So either A = 0 with B = +/- 1 or B = 0 with A = +/- 1. See Figure 16. Since we have divided our full circle of angle 2π into four Cartesian quadrants each must be angle π/2.

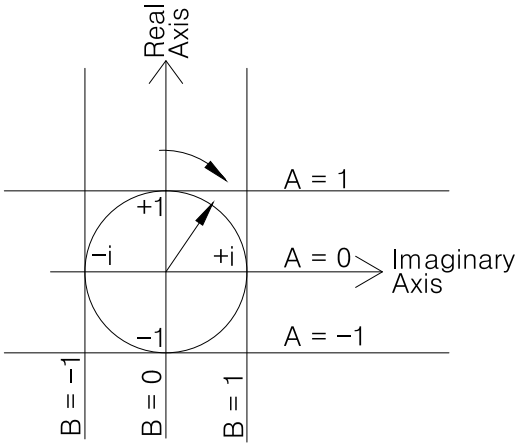

Figure 16 - At the quadrants the A & B values of the complex number A + Bi can be determined by the values of the cartesian plane.

Thus in our π formulation when we recognize +1,+*i*, -1,-*i* as the values for $e^{i\theta}$ at $\theta = 0,\pi/2, \pi, 3\pi/2$, one must realize this is not a backdoor way to introduce a numerical value for π into our equation. The compass point values are a natural consequence of the definition and how the polar equation maps onto the Cartesian coordinate system.

Common Compass Positions We Will Use

If we look at the four values +1, -1, +i, and $-i$ as points on the compass, we see that if we start from one of these points, every time we turn an angle of 2π we return to the starting point:

Starting at $\theta = 0$ the spinner is at +1 (North), and every turn of the spinner by an angle of 2π it will return to this spot, so:

$e^{i0} = +1, \quad e^{i2\pi} = +1, \quad e^{i4\pi} = +1, \quad e^{i6\pi} = +1$

By extension we can see that at $\theta = \pi$ the spinner would point at -1 (South) and would return here again every time an angle of 2π is turned:

$e^{i\pi} = -1, \quad e^{i3\pi} = -1, \quad e^{i5\pi} = -1, \quad e^{i7\pi} = -1$

At $\theta = \pi/2$ the spinner would point at +i (East) and would return here again every time an angle of 2π is turned:

$e^{i\frac{\pi}{2}} = +i, \quad e^{i\frac{5\pi}{2}} = +i, \quad e^{i\frac{9\pi}{2}} = +i, \quad e^{i\frac{13\pi}{2}} = +i$

At $\theta = 3\pi/2$ the spinner would point at -i (West) and would return here again every time an angle of 2π is turned:

$e^{i\frac{3\pi}{2}} = -i, \quad e^{i\frac{7\pi}{2}} = -i, \quad e^{i\frac{11\pi}{2}} = -i, \quad e^{i\frac{15\pi}{2}} = -i$