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The Mechanics of Integers

Part I - A Solution to the Riemann Hypothesis

Michael Ehrlich
PO Box 772393
Steamboat Springs, CO 80477

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Abstract

This paper explores raising an integer n, or its reciprocal, $1/n$, to the complex power s where $s = a + i\alpha$ using the transformations of Modal Harmonics. The Riemann Hypothesis states that all zeros of the sum to infinity of $1/n^s$, also known as the zeta function, occur at $a = ½$. When mapped conventionally the zeta function has no zeros because the real and imaginary parts cannot zero simultaneously. The transformation is an alternate mapping which enables evaluation over a range around a point rather than at that point. With the transformation it is shown that a must equal ½ at all zeros as postulated by Riemann. This analysis shows the deep connection between the set of integers n and the zeros of $1/n^s$. We consider the integers as a special case of the more general number set n^s which includes the complex and irrational numbers. Then α represents the position of a spinner in the complex plane and a determines its magnitude, thus defining whether it is rational or irrational. The zeros of $1/n^s$ occur when 1) the imaginary component of n^s equals zero thus n is real and 2) there are no roots of n thus n is rational. For a single number n^s (not a ratio) this leaves only the set of integers n. Thus at the zeros of $1/n^s$, n^s equals n.

B. A Solution to The Riemann Hypothesis

The Riemann zeta function is the series of integers n raised to the complex variable s:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{6.1}$$

The zeta function has trivial zeros at the even negative integers -2, -4, -6…. The other zeros are solutions of the complex form $s = a + i\alpha$, where a is the real component of s, and $i\alpha$ is the imaginary part. The Riemann Hypothesis states that the non-trivial zeros have real component $a = \frac{1}{2}$. See Reference 1.

I. Single Integers

A. n^S Single Integer / Single Spinner Model

It is assumed that the reader is familiar with the earlier chapters on modal harmonic analysis which are necessary to understand this work. We consider first n^s, using a single integer instead of the sum of integers, and we will work with a single spinner instead of the pair we are accustomed to. We can use the algebraic identity $n^{a+i\alpha} = n^a \cdot n^{i\alpha}$ to separate n^s into two terms, one that is real and one that is complex. For the single integer we will select $n = 2$. Thus $2^{a+i\alpha} = 2^a \cdot 2^{i\alpha}$. The first term is real and represents magnitude. The second term is a complex number of magnitude 1, or, equivalently, a unit spinner since it has a radius of 1. See Figure 60. Raising a real number to an imaginary power i was discussed in Part III of the preceding paper and should be reviewed if necessary.

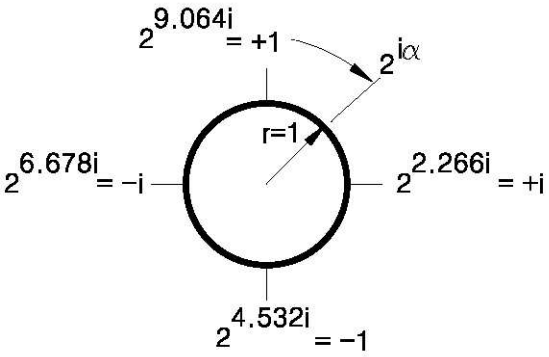


Figure 60 – Single Spinner for $2^{i\alpha}$

In general $2^{i\alpha}$ is complex, but at two positions, spinner down and spinner up, the imaginary component is zero. That is because $2^{4.533i} = -1$ and $2^{9.064i} = +1$. Taking the algebraic inverse of these we obtain:

$$\log_2(-1) = 4.532i \text{ and } 2 \cdot \log_2(-1) = 9.064i$$

The values shown are rounded to three decimals but are in fact irrational. Since the imaginary component is zero at these points, then $2^{i\alpha}$ must be real here. The first value (down) corresponds with the semicircle and has a

value of -1. The second value (up) corresponds with closure and has a value +1. This means that $2^{a+i\alpha}$ is also real at these points since it is the product of the real number 2^a and the unit spinner $2^{i\alpha}$ which is generally a complex number but has a real value at these points.

The unit spinner is cyclical in nature. $2^{i\alpha}$ has an imaginary component with zeros at $\alpha =$ 0, 4.532, 9.064,…or in general zero at $\alpha = ki \cdot \log_2(-1)$, where k is an integer. Since these are spinner up, spinner down, spinner up… the values are +1, -1, +1… Therefore all values for $2^{i\alpha}$ are real at $\alpha = ki \cdot \log_2(-1)$. The cycle repeats at a cycle length of $\lambda = 2i \cdot \log_2(-1)$. See Figure 61. The results for $n = 2$ can be extended to all n. All values of $n^{i\alpha}$ are real at $\alpha = ki \cdot \log_n(-1)$ with a cycle length of $\lambda = 2i \cdot \log_n(-1)$.

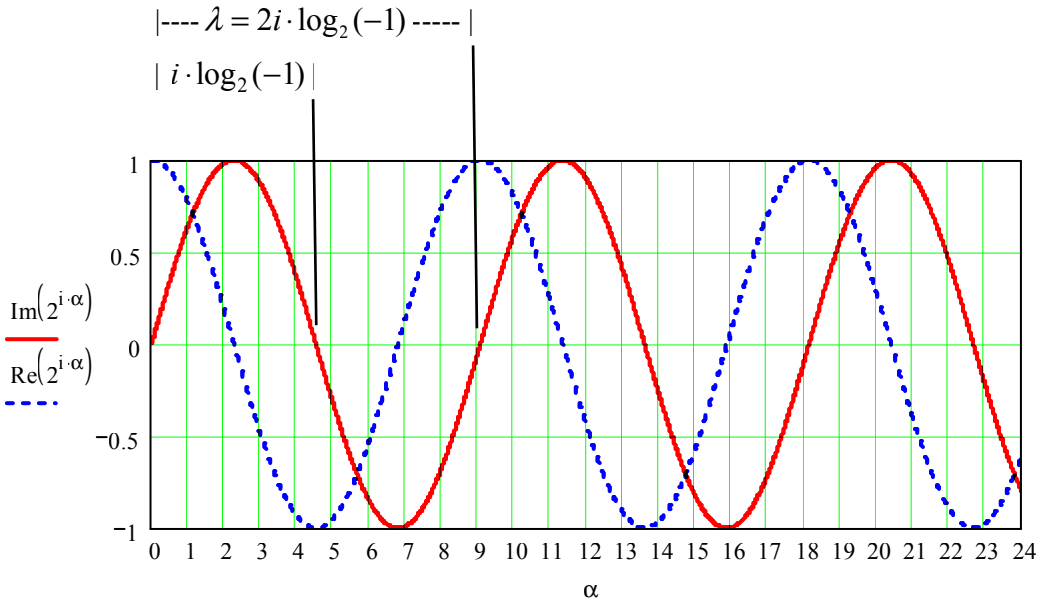


Figure 61 – The Real and Imaginary parts of the unit spinner $2^{i\alpha}$. For the imaginary component, the zeros occur at $ki \cdot \log_2(-1)$ where k is an integer. The first non-trivial imaginary zero occurs at $i \cdot \log_2(-1)$ where $2^{i\alpha} = -1$. Note the cycle repeats at $\lambda = 2i \cdot \log_2(-1)$. At these points the imaginary component is zero and therefore 2^s is real.

B. $1/n^s$ Single Integer / Single Spinner Model

Now we will examine $\frac{1}{n^s}$. Again we are able to substitute $s = a + i\alpha$ and algebraically separate $\frac{1}{n^{a+i\alpha}}$ into $\frac{1}{n^a} \cdot \frac{1}{n^{i\alpha}}$, where $\frac{1}{n^a}$ is a real number representing magnitude, and $\frac{1}{n^{i\alpha}}$ is a complex number of magnitude 1 (a unit spinner). In Figure 62 we graph the imaginary and real parts of the unit spinner $\frac{1}{2^{i\alpha}}$.

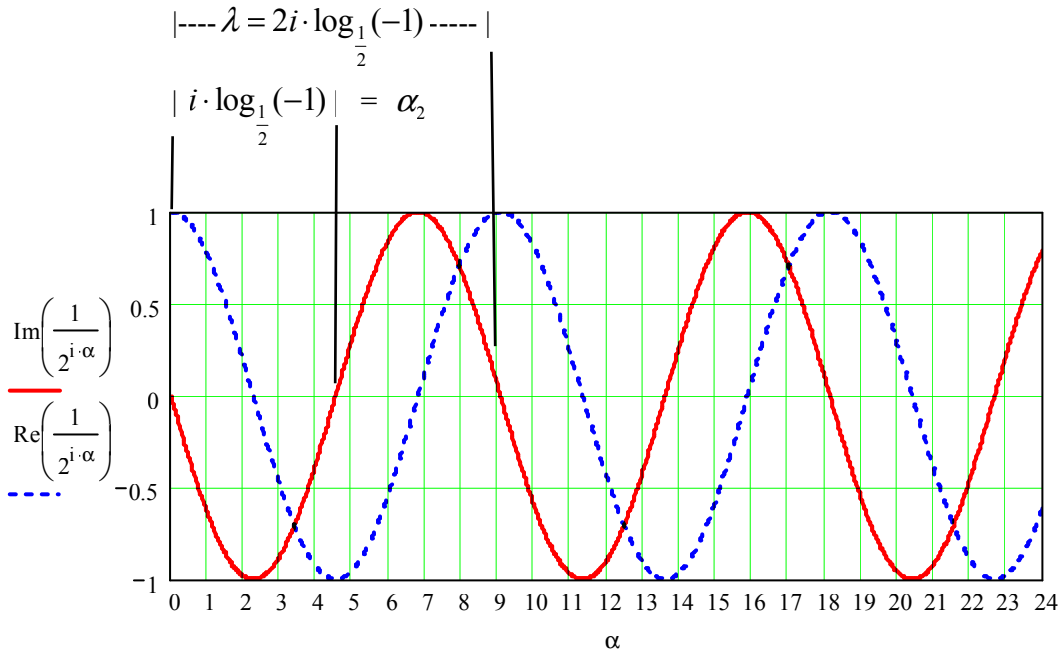


Figure 62 – The Real and Imaginary parts of the unit spinner $\frac{1}{2^{i\alpha}}$. The imaginary zeros occur at $ki \cdot \log_{\frac{1}{2}}(-1)$ with k an integer. The first non-trivial imaginary zero occurs at $\alpha_2 = i \cdot \log_{\frac{1}{2}}(-1)$ where $\frac{1}{2^{i\alpha}} = -1$. Note the cycle repeats at $\lambda = 2ki \cdot \log_{\frac{1}{2}}(-1)$. At these points the imaginary component is zero and $\frac{1}{2^s}$ is real.

There is specific value of α called α_n where the spinner points down and $\frac{1}{n^{i\alpha}}$ is real with a value of -1. Then $\frac{1}{n^s}$ is also real since it is the product of two real numbers: $\frac{1}{n^a}$ which is always a real number, and $\frac{1}{n^{i\alpha}}$ which equals -1 at $\alpha = \alpha_n$. For any n, at $\alpha_n = i \cdot \log_{\frac{1}{n}}(-1)$ the spinner points down, the imaginary component of $\frac{1}{n^{i\alpha}}$ is zero, and a real value is obtained for $\frac{1}{n^s}$. So at α_n, $\frac{1}{n^{i\alpha}} = -1$, and $\frac{1}{n^s}$ is a real number given by $\frac{1}{n^s} = -\frac{1}{n^a}$. Thus for any $\frac{1}{n^s}$, the value of α_n is determined by:

$$\alpha_n = i \cdot \log_{\frac{1}{n}}(-1) \tag{6.2}$$

So for any n, when $\alpha = \alpha_n$ a real value is obtained for $\frac{1}{n^s}$. Figure 63 shows the α_n values for $n = 2, 3, 4 \ldots 12$.

$$\alpha_{(n)} := i \cdot \log\left(-1, \frac{1}{n}\right)$$

$n =$	$\alpha_n =$	$\frac{1}{\alpha_n} =$
2	4.532	0.2206
3	2.86	0.3497
4	2.266	0.4413
5	1.952	0.5123
6	1.753	0.5703
7	1.614	0.6194
8	1.511	0.6619
9	1.43	0.6994
10	1.364	0.7329
11	1.31	0.7633
12	1.264	0.7910
13	1.225	0.8164
14	1.19	0.8400
15	1.16	0.8620
16	1.133	0.8825

Figure 63 - Values for α_n and $\frac{1}{\alpha_n}$ for $\frac{1}{n^s}$. Note that $\frac{1}{\alpha_2}+\frac{1}{\alpha_2}=\frac{1}{\alpha_4}$ and $\frac{1}{\alpha_2}+\frac{1}{\alpha_3}=\frac{1}{\alpha_6}$.

The equation has no solution at $n=1$. That is because it asks for a result of -1 by raising 1 to an unknown power. However 1 raised to any power is always 1 so there is no solution at $n=1$.

Note that Composite Numbers, which in real space are formed by the multiplication of Primes, are formed by the addition of the $\frac{1}{\alpha_n}$ in this space. For example, $\frac{1}{\alpha_2}+\frac{1}{\alpha_3}=\frac{1}{\alpha_6}$. The Composite Numbers can only be formed by one combination of primes and no more, per the Fundamental Rule of Arithmetic. For example the composite 1080 can only be made from the primes $2^3 \cdot 3^3 \cdot 5$ and no other combination.

By extension the reciprocal alpha $\frac{1}{\alpha_{1080}}$ can only be made from the reciprocal alphas $\frac{3}{\alpha_2}+\frac{3}{\alpha_3}+\frac{1}{\alpha_5}$ and no other combination. Since the $\frac{1}{\alpha_n}$ add instead of multiply, there is only one way to add the individual reciprocal alpha primes $\frac{1}{\alpha_P}$ to get any specific reciprocal alpha composite $\frac{1}{\alpha_C}$.

Thus the $\frac{1}{\alpha_P}$ can be viewed as fundamental building blocks and combine through addition to create the $\frac{1}{\alpha_C}$ of the composites similar to the way elements add to make compounds. They also can combine in only one way, similar to the way a chemical compound can only have one combination of elements.

Each $\frac{1}{\alpha_P}$ is unique and corresponds to an imaginary zero of $\frac{1}{n_P^{\,s}}$ where n_P is a prime number. Additionally each possible combination of these $\frac{1}{\alpha_P}$ results in a unique composite number with a unique $\frac{1}{\alpha_C}$ that also corresponds to an imaginary zero for $\frac{1}{n_C^{\,s}}$ where n_C is a composite number. So in general each reciprocal alpha n, $\frac{1}{\alpha_n}$, is unique and corresponds to an imaginary zero for $\frac{1}{n^s}$.

The reason for this is that all members of the set *n* must have an imaginary component of zero since the integers are real numbers. At $\alpha = \alpha_n$ and its multiples the spinner position is either up or down and the imaginary component is zero so the number is real.

C. Real and Imaginary Zeros

Returning to Figures 61 and 62, we see that in both cases the real and imaginary components are out of phase and *never* zero at the same time. Therefore is not possible for $\frac{1}{n^s}$ to zero in the traditional sense. Recall that $\frac{1}{n^s} = \frac{1}{n^{a+i\alpha}} = \frac{1}{n^a} \cdot \frac{1}{n^{i\alpha}}$, and $\frac{1}{n^a} \cdot \frac{1}{n^{i\alpha}}$ is the real number $\frac{1}{n^a}$ times the complex unit spinner $\frac{1}{n^{i\alpha}}$. It is this unit spinner we are interested in, which has the imaginary and real components we are discussing. $\frac{1}{n^a}$ cannot be equal to zero as this is undefined, and the imaginary and real components of $\frac{1}{n^{i\alpha}}$ do not zero simultaneously. Therefore the imaginary and real components of $\frac{1}{n^s}$ cannot zero simultaneously either, if mapped in the conventional way.

D. Significance of Zeros

What is the significance of obtaining a zero value for $\frac{1}{n^s}$? We consider the real numbers as a subset of the complex numbers and note how we factored $\frac{1}{n^s}$ into the real part or magnitude $\frac{1}{n^a}$ and complex unit spinner $\frac{1}{n^{i\alpha}}$.

Then any real number can be seen as a special case of a more general number set that includes the complex (real plus imaginary) numbers. This more general number set would involve a real number times a complex unit spinner. In the special case where we have a real number the unit spinner would be up or down and its imaginary part equal to zero. This is the case for all real numbers.

So in the special case of $\alpha = \alpha_n$ and its multiples, $\frac{1}{n^{i\alpha}} = \pm 1$ and $\frac{1}{n^s}$ is a real number. Thus α controls the complex part of this more general number system, and the imaginary part of $\frac{1}{n^{i\alpha}}$ (and $\frac{1}{n^s}$) must be zero at all real numbers which includes the set of integers.

This is all we can see in this mapping space but it is only half of the story. We will need a different mapping space to appreciate the second set of mechanics.

Specifically, with $s = a + i\alpha$, α determines the position of a complex unit spinner in the complex plane, and a determines the magnitude of this spinner which is a real number, but not necessarily rational. The number may in fact be a root of n and be irrational. It is the value of a which determines whether n^s is a rational number or an irrational root. Thus n^s represents a number in the more general number set that includes both the complex numbers and the real but irrational numbers. When a = ½ there are no irrational roots and the number is real.

For $\frac{1}{n^s}$ or its sum to equal zero, both the imaginary and real parts of $\frac{1}{n^s}$ must be zero. The transformation enables the mapping of $\frac{1}{n^s}$ and its sum over an action $A \to B$ rather than a single point. In this mapping space it is possible for both the imaginary and real parts to zero simultaneously.

After transformation the real part of $\frac{1}{n^s}$ equals zero when $a = 1/2$. Additionally, as we saw that α controlled the more general number set which included imaginary numbers that needed to zero at all integers, a controls the

more general number set containing the irrational roots of n $\left(example: \frac{1}{n^{1/3}} = \frac{1}{\sqrt[3]{n}} \right)$. These irrational components must also zero at the integers, and this is the meaning of real part of $\frac{1}{n^s}$ zeroing, that it does not contain any irrational roots of *n*.

We will see how this relates to the integers. At the integers both the imaginary and real parts of $\frac{1}{n^s}$ are equal to zero. This corresponds to both the complex spinner pointing up or down so there is no imaginary component, and to there being no irrational roots so the number is real.

Thus obtaining a zero for $\frac{1}{n^s}$ implies that the imaginary part of $\frac{1}{n^s}$ is zero and there are no irrational roots. This limits us to the set of real and rational numbers. Since we are dealing with a single number and not a ratio, the only numbers remaining are the integers. This is the significance of finding zeros for $\frac{1}{n^s}$. In looking at the more general number set, the integers are a special case where both the imaginary component is zero (a real number) and there are no roots (a rational number).

So $\frac{1}{n^s}$ with $s = a + i\alpha$ represents a general number with three controls. n is the base integer, α is the position of the unit spinner in the complex plane, and a controls the irrational roots. When $\alpha = \alpha_n$ the imaginary part is zero, and when $a = 1/2$ there are no irrational roots. Thus zeroing of the general number $\frac{1}{n^s}$ restricts us to real and rational numbers only. Ratios are not included so we have only integers remaining. This is the deep connection between the integers and the zeros of $\frac{1}{n^s}$ which has proven so elusive. The integers occur are the exact spots in the general number set where the imaginary component is zero and there are no irrational roots.

II. Transformations of Single Integers

A. Riemann Functional Equation

Riemann extended the zeta function over the complex plane using a functional equation that has only a single pole at $s = 1$ (See Reference 2):

$$\pi^{-s/2} \cdot \Gamma(\frac{s}{2}) \cdot \zeta(\mathrm{s}) \; = \; \pi^{-(1-s)/2} \cdot \Gamma(\frac{1-s}{2}) \cdot \zeta(1-\mathrm{s}) \tag{6.3}$$

This can be rewritten as

$$\pi^{-s/2} \cdot \Gamma(\frac{s}{2}) \cdot \zeta(\mathrm{s}) \; - \; \pi^{-(1-s)/2} \cdot \Gamma(\frac{1-s}{2}) \cdot \zeta(1-\mathrm{s}) \; = 0 \tag{6.3A}$$

Or with dummy variable Θ: $$\pi^{-\Theta/2} \cdot \Gamma(\frac{\Theta}{2}) \cdot \zeta(\Theta)\Big|_{1-\mathrm{s}}^{s} = 0 \tag{6.3B}$$

Which is of the form $$\int_{1-s}^{s} F(\Theta) \cdot d\Theta \; = \; 0 \tag{6.3C}$$

We see the functional equation is actually an integral equation evaluated at lower and upper bounds $A = 1 - s$ and $B = s$. We saw this condition arise from the General Transformation Equation (GTE) where we integrated between two points A and B to map an action in a harmonic space:

$$\int_{A}^{B} f(\theta) \cdot d\theta = \int_{A}^{B} F(\Theta) \cdot d\Theta \tag{6.4C}$$

It appears that Riemann was using a transformation to study the underlying equation $\zeta(\mathrm{s})$.

B. $1/n^{s}$ Single Integer / Two Opposing Spinners

We too will use a transformation to study the equation $\zeta(\mathrm{s})$ over an interval, but first we will first look at a the case of a single value of *n* for $\frac{1}{n^{s}}$ rather than a summation of *n* values. The boundary conditions we will use are $A = 1 - s$ and $B = s$, to be explained later. So under transformation from one mapping space to another $1/n^{s}$ will be evaluated at two points, s and $1 - s$ and we take the difference between these two values.

We substitute $s = a + i\alpha$ into $1/n^{s}$. The terms can then be separated into real and complex components using the identity $\frac{1}{n^{a+i\alpha}} = \frac{1}{n^{a}} \cdot \frac{1}{n^{i\alpha}}$:

$$\frac{1}{n^{s}}\Big|_{1-s}^{s} \; = \; \frac{1}{n^{s}} - \frac{1}{n^{(1-s)}} \; = \; \frac{1}{n^{(a+i\alpha)}} - \frac{1}{n^{(1-(a+i\alpha))}} \; = \left[\frac{1}{n^{a}} \cdot \frac{1}{n^{+i\cdot\alpha}}\right] - \left[\frac{1}{n^{(1-a)}} \cdot \frac{1}{n^{-i\cdot\alpha}}\right] \tag{6.5}$$

Note how this separates each of the two terms that are subtracted into the product of a real component and a complex component. The expression $\frac{1}{n^s} - \frac{1}{n^{(1-s)}}$ can be modeled as the difference of two opposing spinners. Each spinner has a real term which represents its amplitude. These are $\frac{1}{n^a}$ and $\frac{1}{n^{(1-a)}}$, respectively. Each spinner also has a complex term of magnitude 1 (a unit spinner) representing its phase. These are $\frac{1}{n^{+i\alpha}}$ and $\frac{1}{n^{-i\alpha}}$, respectively. The real and complex terms for each spinner are multiplied. See Figure 64.

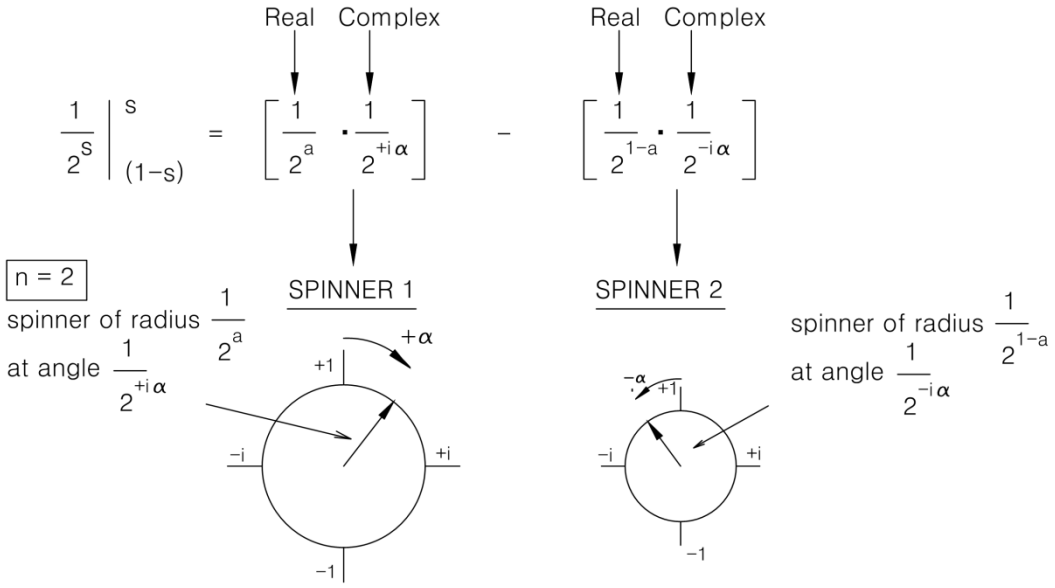


Figure 64 – The difference of two opposing spinners is used to model $\frac{1}{n^s}\Big|_{1-s}^{s}$. Each spinner has a real factor which determines the radius and a complex factor which determines its position. The complex factor is itself a spinner of magnitude 1 (a unit spinner).

We will show that in order for a non trivial zero to occur, two things must happen. First, if we look at just the complex term from each of the two spinners, the difference between these complex terms must be zero. If it is not then $\frac{1}{n^s}\Big|_{1-s}^{s}$ will have a complex component, so this will not be a zero. Second, the difference between the two real components must also be zero.

II.B.1. The Difference of The Complex Component of Each of The Two Spinners Must be Zero

Figure 63 shows the model with the two opposing spinners $\frac{1}{n^{+i\alpha}}$ and $\frac{1}{n^{-i\alpha}}$. We see that these are symmetric about zero and therefore the difference of these terms will always result in the real component being zero.

To put numbers to this we will look at $n = 2$ as an example. We will use $\alpha = 3$ for a randomly chosen value that ***does not*** result in a zero. When we take the difference between the complex terms $\left[\frac{1}{n^{+i\alpha}}\right] - \left[\frac{1}{n^{-i\alpha}}\right]$ we see that the real component always subtracts out to zero regardless of the value of α. Thus at any value of α the subtraction of these terms has a real component equal to zero:

$\left[\frac{1}{2^{(i\cdot 3)}}\right] = -0.487 - 0.873i$ $\left[\frac{1}{2^{(-i\cdot 3)}}\right] = -0.487 + 0.873i$ $\left[\frac{1}{2^{(i\cdot 3)}}\right] - \left[\frac{1}{2^{(-i\cdot 3)}}\right] = -1.747i$

Spinner 1 Spinner 2 Difference

In general the imaginary part is not equal to zero. However at the value $\alpha_2 = i \cdot \log_{\frac{1}{2}}(-1) = 4.532$ the spinners $\frac{1}{2^{+i\alpha_2}}$ and $\frac{1}{2^{-i\alpha_2}}$ both point directly down and have value -1. This causes the imaginary part to subtract out as well. So at this value both the real and the imaginary components of the difference of the complex terms $\left[\frac{1}{2^{+i\alpha_2}}\right] - \left[\frac{1}{2^{-i\alpha_2}}\right]$ subtract out to zero:

$\alpha_2 := i\cdot\log\left(-1, \frac{1}{2}\right)$ $\alpha_2 = 4.532$

$\frac{1}{2^{(i\cdot\alpha_2)}} = -1$ $\frac{1}{2^{(-i\cdot\alpha_2)}} = -1$ $\frac{1}{2^{(i\cdot\alpha_2)}} - \frac{1}{2^{(-i\cdot\alpha_2)}} = 0$

Spinner 1 Spinner 2 Difference

Figures 65 & 66 show spinner models for these numerical examples.

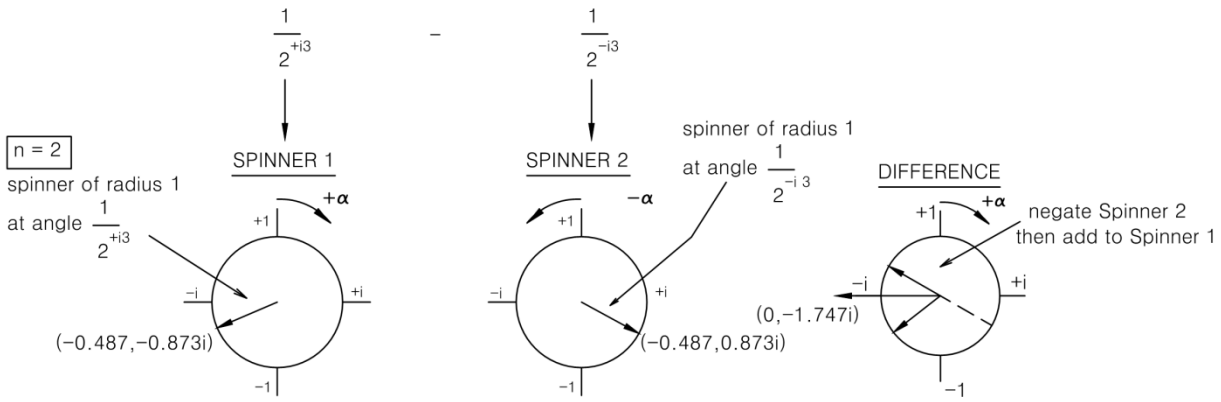


Figure 65 – **The difference of the complex unit spinners has a real component of zero but the imaginary** component is generally not zero.

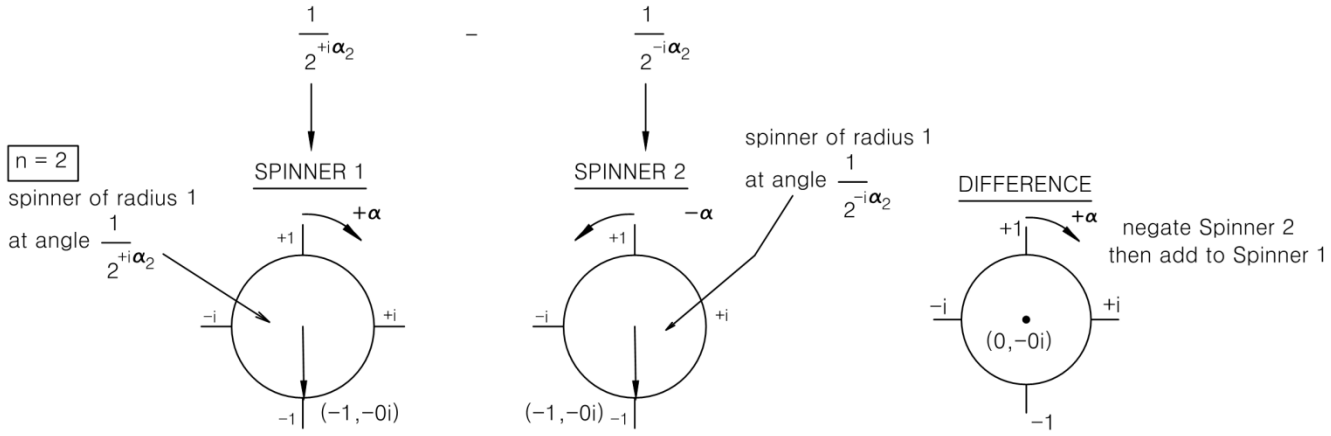


Figure 66 – At $\alpha = \alpha_2$ **both spinners point downwards and the imaginary component cancels out as well** resulting in a zero.

In Figure 67 we graph the **difference of the complex unit spinners** $\left[\frac{1}{2^{+i\alpha_2}}\right] - \left[\frac{1}{2^{-i\alpha_2}}\right]$. **If we let** α_{2k} represent ***the set of zeros***, we see **that the zeros occur at** $\alpha_{2k} = ki \cdot \log_{\frac{1}{2}}(-1)$, where k is an integer. **These zeros** correspond with **both spinners pointing down** (-1). The first non-trivial zero is $\alpha_2 = i \cdot \log_{\frac{1}{2}}(-1)$. The length of each cycle is $\lambda_2 = 2i \cdot \log_{\frac{1}{2}}(-1)$.

$$F_2(\alpha) := \left(\frac{1}{2^{i\cdot\alpha}} - \frac{1}{2^{-i\cdot\alpha}} \right)$$

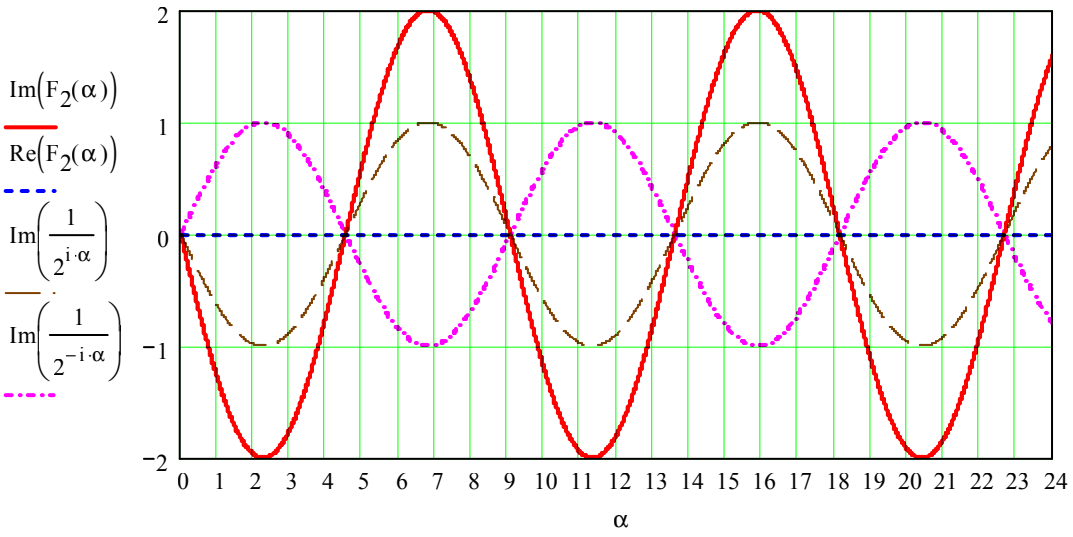


Figure 67: $\left[\frac{1}{2^{+i\alpha_2}}\right]-\left[\frac{1}{2^{-i\alpha_2}}\right]$ is the complex component of $\frac{1}{2^s}$ evaluated at $A = 1-s$ and $B = s$, which can be modeled as the difference of two opposing spinners. The real part always subtracts out to zero. The imaginary part has zeros at $\alpha_{2k} = ki\cdot\log_{\frac{1}{2}}(-1)$ where k is an integer. The cycle length is $\lambda_2 = 2i\cdot\log_{\frac{1}{2}}(-1)$ and the first non-trivial zero is $\alpha_2 = i\cdot\log_{\frac{1}{2}}(-1)$.

Generalizing, for each integer n there is a value for α known as α_n that will cause the subtraction of the two complex phase terms to cancel out:

$$\left[\frac{1}{n^{+i\cdot\alpha_n}}\right]-\left[\frac{1}{n^{-i\cdot\alpha_n}}\right] = 0 \qquad (6.6)$$

A solution to this is $$\alpha_n = i\cdot\log_{\frac{1}{n}}(-1) \qquad (6.7)$$

Or in general $$\alpha_{nk} = ki\cdot\log_{\frac{1}{n}}(-1) \qquad (6.8)$$

This is true because $\alpha_n = i\cdot\log_{\frac{1}{n}}(-1)$ both spinners are half way around the circle. So the two opposing spinners of (6.6) are both equal to -1 there.

The α_n here are the same as the α_n values shown in Figure 62 because the formula $\alpha_n = i \cdot \log_{\frac{1}{n}}(-1)$ is the same for both models.

The set of zeros for *n* are the multiples of α_n, or $\alpha_{nk} = ki \cdot \log_{\frac{1}{n}}(-1)$ where k is integer and α_{nk} is the k^{th} multiple of α_n. We can expand the number α_n in (6.6) to the set α_{nk} :

$$\left[\frac{1}{n^{+i\alpha_{nk}}}\right] - \left[\frac{1}{n^{-i\alpha_{nk}}}\right] = 0 \tag{6.9}$$

As a specific example we see that for $n = 2$ any of the α_{2k} set will result in zero.

$$\left[\frac{1}{2^{+i\alpha_{2k}}}\right] - \left[\frac{1}{2^{-i\alpha_{2k}}}\right] = 0 \tag{6.10}$$

II.B.2. The Difference of The Real Components of The Two Spinners Must be Zero

When $\alpha = \alpha_n$ the value of $\left[\frac{1}{n^{+i\alpha}}\right]$ and $\left[\frac{1}{n^{-i\alpha}}\right]$ are both -1. This causes the complex part to drop out of $\frac{1}{n^s}$.

To demonstrate this we substitute this into $\left[\frac{1}{n^a} \cdot \frac{1}{n^{+i\alpha}}\right] - \left[\frac{1}{n^{(1-a)}} \cdot \frac{1}{n^{-i\alpha}}\right]$ to get $\left[\frac{1}{n^a} \cdot (-1)\right] - \left[\frac{1}{n^{(1-a)}} \cdot (-1)\right]$

which equals $$\left[\frac{-1}{n^a}\right] - \left[\frac{-1}{n^{(1-a)}}\right] \tag{6.11}$$

This is the difference of two real numbers and is therefore real. We will only get zero for this expression when:

$$a - (1-a) = 0 \tag{6.12}$$

This is only true when $a = 1/2$.

Therefore all non-trivial zeros of $\frac{1}{n^s}\Big|_{1-s}^{s}$ have:

1) $\left[\frac{1}{n^{+i\alpha_{nk}}}\right] - \left[\frac{1}{n^{-i\alpha_{nk}}}\right] = 0$ (difference of complex components equals zero) with solution set $\alpha_{nk} = ki \cdot \log_{\frac{1}{n}}(-1)$, and

2) $a - (1-a) = 0$ (difference of real components equals zero) with solution $a = 1/2$.

C. Selection of Boundary Values for Transformation

The two conditions just stated establish a rule set which determines the boundary values $A = s$ and $B = 1 - s$ for our integral.

In order for $\left[\frac{1}{n^{+i\alpha_{nk}}}\right] - \left[\frac{1}{n^{-i\alpha_{nk}}}\right] = 0$ to be true then $[+i\alpha_{nk}]$ must equal $[-i\alpha_{nk}]$, or equivalently $[+i\alpha_{nk}] - [-i\alpha_{nk}] = 0$. This is the imaginary part of the rule set.

In order for $[a] - [1-a] = 0$ to be true then $[a]$ must equal $[1-a]$. This is the real part of the rule set.

Combining the components we obtain $[a + i\alpha_{nk}]$ = $[(1-a) - i\alpha_{nk}]$ or $[a + i\alpha_{nk}] - [(1-a) - i\alpha_{nk}] = 0$.
Note that the value for α that results in an imaginary zero, $\alpha = \alpha_{nk}$ is used.

We recognize $[a + i\alpha_{nk}]$ as $[s]$. Also we recognize $[(1-a) - i\alpha_{nk}]$ as $[1-s]$.
These are the boundaries for our integral, $A = [s]$ and $B = [1-s]$.

Thus our two conditions:

1) the difference of the complex components equals zero, expressed mathematically as $i\alpha_{nk} - (-i\alpha_{nk}) = 0$, and
2) the difference of the real components equals zero, expressed mathematically as $a - (1-a) = 0$,

establish the imaginary and real components of the rule set:
1) $[+i\alpha_{nk}] - [-i\alpha_{nk}] = 0$ and
2) $[a + i\alpha_{nk}] - [(1-a) - i\alpha_{nk}] = 0$, respectively.

The combination of the real and imaginary parts of the rule set define the boundary conditions *A* and *B*. Thus our boundary conditions are an expression of the rules that we require the mechanics to follow.

We will see that these are the requirement which distinguish the integers from the general number set. The boundary values contain the special requirements that the integers must be real and rational.

We considered using $A = s/2$ and $B = (1-s)/2$ as the boundary conditions. This would be more consistent with other transformations, like the Fibonacci, and as we noted in Figure 78, the difference between the spinners needs to be divided by 2 for normalization to the unit spinner. When the analysis is executed that way some equations would carry factors of 2 or ½, but the results of the analysis are essentially the same. However when a= ½, the exponent appears as ¼, or the fourth root of *n* which looks more complicated than needed. So long as normalization is recognized it is personal preference. We chose Riemann's traditional $A = s$ and $B = 1 - s$.

D. $n = 3$

Now we will examine $n = 3$. Here $\alpha_3 = i \cdot \log_{\frac{1}{3}}(-1)$. At this value and its multiples the two spinners point down or up only and the imaginary component is zero. ***That is the only way this can occur. All other values for α will produce complex numbers.***

So at α_3, the imaginary terms cancel out of $\left[\frac{1}{3^a}\cdot\frac{1}{3^{+i\cdot\alpha_3}}\right]-\left[\frac{1}{3^{(1-a)}}\cdot\frac{1}{3^{-i\cdot\alpha_3}}\right]$ and it reduces to $\left[\frac{1}{3^a}\right]-\left[\frac{1}{3^{(1-a)}}\right]$. In order for this to subtract to zero $\left[\frac{1}{3^a}\right]$ must equal $\left[\frac{1}{3^{(1-a)}}\right]$. This equals zero only $a = 1/2$. ***That is the only way this can occur. All other values of "a" produce irrational numbers.***

This is true in general for all n.

III. A. Formation of Composite Numbers from Primes

Now we will look at how the composite numbers are formed from the prime numbers But instead of using the variable α, here we will use its reciprocal, $1/\alpha_n$ or α_n^{-1}. We just looked at α_2 and α_3 which are the values of α which cause the imaginary part of $\left[\frac{1}{2^a}\right]-\left[\frac{1}{2^{(1-a)}}\right]$ and $\left[\frac{1}{3^a}\right]-\left[\frac{1}{3^{(1-a)}}\right]$ to zero, respectively. Reprinting our table:

$\alpha_{(n)} := i\cdot\log\left(-1, \frac{1}{n}\right)$

n =	α_n =	$\frac{1}{\alpha_n}$ =
2	4.532	0.2206
3	2.86	0.3497
4	2.266	0.4413
5	1.952	0.5123
6	1.753	0.5703
7	1.614	0.6194
8	1.511	0.6619
9	1.43	0.6994
10	1.364	0.7329
11	1.31	0.7633
12	1.264	0.7910
13	1.225	0.8164
14	1.19	0.8400
15	1.16	0.8620
16	1.133	0.8825

Note that $1/\alpha_4 = 1/\alpha_2 + 1/\alpha_2$ and $1/\alpha_6 = 1/\alpha_2 + 1/\alpha_3$.

This relationship is true in general. The reciprocal alpha of each composite number is the sum of the reciprocal alphas of the primes that compose it. The integers themselves have a product relationship between primes and composites, but the reciprocal alpha n's add because of the logarithmic relation between n and α_n.

Thus the $1/\alpha_n$'s of the prime numbers add to form the $1/\alpha_n$ of any composite, or $1/\alpha_C = \sum 1/\alpha_P$ where C and P represent the composite and prime numbers, respectively.

Note that in real space the primes multiply to form composites but in the transformed space the reciprocal alpha n's of the primes add to form the reciprocal alpha n's of the composite numbers. Recall these reciprocal alpha n's are the value for α that results in the imaginary part of $1/n^s$ to equal zero, where $s = a + i\alpha$.

Since the prime numbers cannot be formed through multiplication of any other numbers, it follows that the $1/\alpha_P$ corresponding with any prime numbers cannot be formed by the addition of any other $1/\alpha_P$. All of the prime numbers are unique with zeros at $\alpha_{Pk} = ki \cdot \log_{\frac{1}{P}}(-1)$ where k is an integer and P is a prime number. Also note that all of the α_n and their reciprocals, both prime and composite, are irrational.

In real space each composite can only be formed through multiplying one and only one combination of primes as per the Fundamental Theory of Arithmetic - that is there is only one set of primes that multiply to make each composite. Thus the reciprocal alpha *n* for each composite number is formed by adding one and only one combination of reciprocal alpha *n*'s corresponding to the underlying prime numbers.

The reciprocal alpha *n*'s of the primes add to form those of the composites because the phase relationship $\frac{1}{n^{i\alpha}}$ has a logarithmic solution. The logarithmic function effectively transforms multiplication into addition. We stress this because it is a more basic and intuitive operation than multiplication. For example, the assembly of elements into molecules is more easily visualized. Two atoms of the element hydrogen combine with one atom of element oxygen to form H_2O, and this molecule can only be formed in one way.

So in real space, for example, the composites:

. $6 = 2 \cdot 3$ which transforms to $1/\alpha_2 + 1/\alpha_3$ and

$216 = 2^3 \cdot 3 \cdot 7$ which transforms to $3 \cdot \left(1/\alpha_2\right) + 1/\alpha_3 + 1/\alpha_7$

This is a simple linear relationship of the form:

$$1/\alpha_C = j_2 \cdot \left(1/\alpha_2\right) + j_3 \cdot \left(1/\alpha_3\right) + j_5 \cdot \left(1/\alpha_5\right) + j_7 \cdot \left(1/\alpha_7\right) + \ldots\ldots \qquad (6.13)$$

where j_n is the number of times a specific prime number is a multiple (power), which equals to the number of times we need to add the reciprocal alpha *n* of that prime to form the reciprocal alpha *n* of the composite $1/\alpha_C$.

Figure 68 is a graph of the $1/\alpha_n$ for each *n*. We see that the $1/\alpha_n$ values for 2 and 3 add to form the value for 6, and 3 times the value for 2 is the value for 8 since $2^3 = 8$.

The $1/\alpha_C$ for each composite is the sum of the $1/\alpha_P$ of its underlying primes. The $1/\alpha_P$ are irrational, and each prime has a different logarithmic base so these do not combine. Therefore each $1/\alpha_C$ is also irrational and contains "fingerprints" of the $1/\alpha_P$ that compose it. The coefficients j_n in (6.13) create linear combinations of the $1/\alpha_P$ but each $1/\alpha_P$ is of a different irrational base that acts like a signature. The irrational base of each $1/\alpha_P$ will remain as separate components under linear combination.

In the case of only one non-zero j_n, then only one $1/\alpha_P$ contributes to the $1/\alpha_C$. This occurs when the composite is a power of one prime only such as $2^3 = 8,$ and in this case only there is only one base, $1/\alpha_2$. But in general the linear combination will involve different $1/\alpha_P$.

$$\alpha_n(n) := i \cdot \log\left(-1, \frac{1}{n}\right)$$

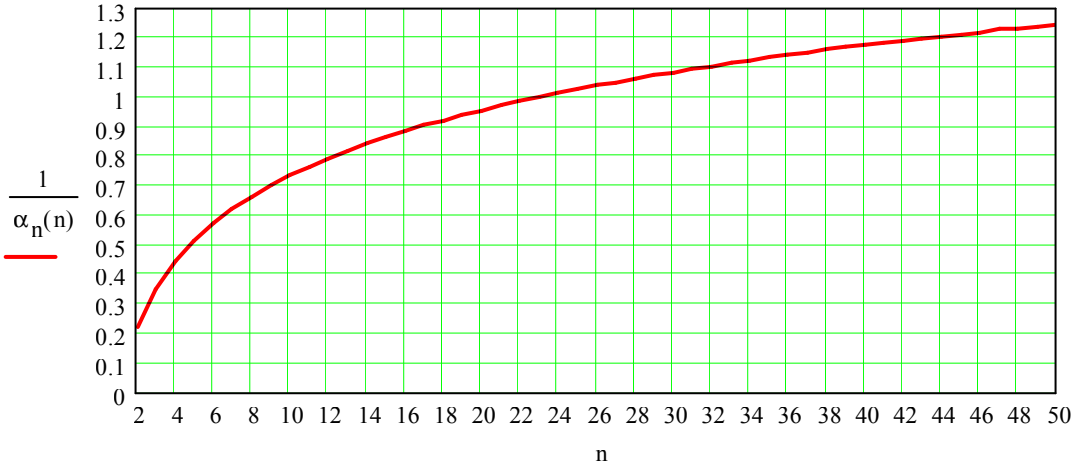


Figure 68 – The values of $1/\alpha_n$ for each n. The values add. For example $1/\alpha_2 + 1/\alpha_3 = 1/\alpha_6$ and $3 \cdot \left(1/\alpha_2\right) = 1/\alpha_8$. α_n is the value for α that results in the imaginary part of $\frac{1}{n^s}\Big|_{(1-s)}^{s}$ to be zero.

Returning to the two opposing spinner model, the $1/\alpha_C$ value corresponds to one pair of spinners, both pointing down. This is a single pair representing the composite. We can equate this to an assembly of spinner pairs, where one pair is contributed from each underlying prime. Each of these spinner pairs points down at its own $1/\alpha_P$,

and these $1/\alpha_P$, sum to form $1/\alpha_C$. The summing is inverse like resistors in parallel. The relation for equating these spinner positions is $1/\alpha_C = \sum 1/\alpha_P$.

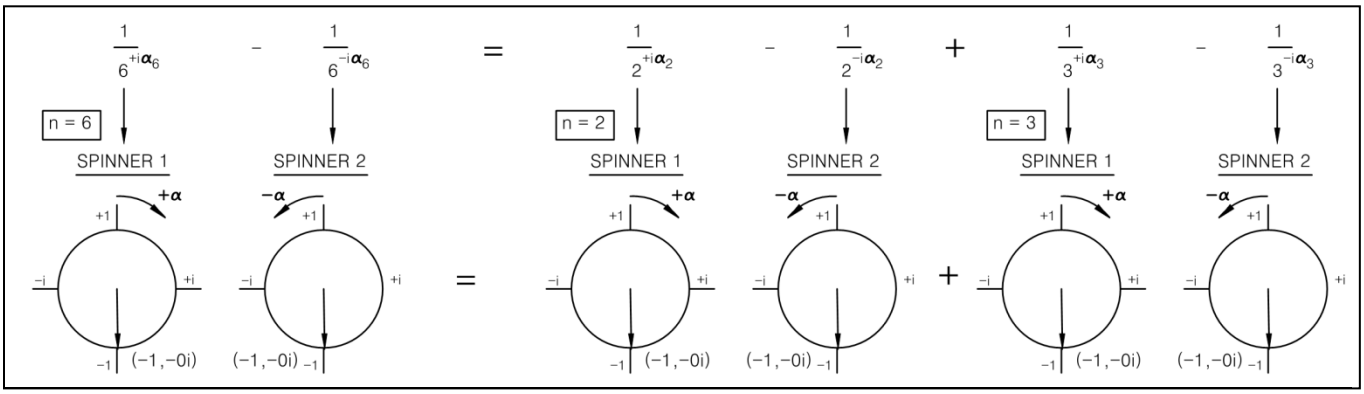


Figure 69 – The spinners of the pair for the composite 6 point down at α_6. This pair can be equated to the sum of a pair of spinners for each prime number that makes up the composite. Each pair of primes must zero independently and points down at its own α_P value. The reciprocal alphas of the primes add to equal the reciprocal alpha of the composite.

B. Summing of Reciprocal Alpha n's

For a single n we saw that the specific value α_n caused the imaginary components to subtract out of $\left[\frac{1}{n^a}\cdot\frac{1}{n^{+i\alpha_n}}\right]-\left[\frac{1}{n^{(1-a)}}\cdot\frac{1}{n^{-i\alpha_n}}\right]$ leaving $\left[\frac{1}{n^a}\right]-\left[\frac{1}{n^{(1-a)}}\right]$ which involves only real numbers. This in turn has a specific value of $a = 1/2$ which causes the real component to subtract out creating a zero.

For example we use the composite $n = 6$, $1/\alpha_6 = 1/\alpha_2 + 1/\alpha_3$ and $\alpha_6 = 1.753...$ There is a spinner pair for the composite and the difference of these two spinners will have an imaginary part equal to zero at $\alpha = \alpha_6$ (spinner down).

The spinner pair for the composite can be equated to the sum of the spinner pairs of the underlying primes $n = 2$ and $n = 3$. The $n = 2$ pair will have an imaginary zero at α_2 and the $n = 3$ at α_3. The reciprocals of these α_P add to exactly the reciprocal of α_C. ***Note that each pair of primes must zero independently.*** That is each spinner must subtract to zero with its mate.

That is because every prime number has a unique reciprocal alpha that is irrational of and of a different base than all the others, so it will never cancel with another unique number that is irrational and of a different base.

In general, no matter how large the n, or equivalently, how small the amounts being added are, if each pair does not subtract out to zero with its counterpart then there will be an imaginary component in the sum because each α_n is of a different irrational base and will always remain separate under linear combination.

Additionally the signature of each underlying prime is present through its reciprocal alphas since these are irrational and of different bases and will not combine under the addition of (6.13).

When an α value occurs that corresponds to a linear combination of spinner pairs where each pair subtracts out to zero, then the imaginary component of the sum is zero as there are only real numbers to work with.

The real terms $\left[\frac{1}{n^a}\right]-\left[\frac{1}{n^{(1-a)}}\right]$ also must zero pair by pair for the sum to zero. That is because these are irrational numbers. For example, $1/\sqrt{2}$ terms can only subtract from $1/\sqrt{2}$ terms and $1/\sqrt{3}$ terms must subtract form $1/\sqrt{3}$ terms for there to be a zero. No matter how large the n or how small the amounts being added, a root of one base will never zero with a root from another base.

Also, in order for $\left[\frac{1}{n^a}\right]-\left[\frac{1}{n^{(1-a)}}\right]$ to subtract to zero, $\left[\frac{1}{n^a}\right]$ must equal $\left[\frac{1}{n^{(1-a)}}\right]$ for each n. This means that a must equal $(1-a)$. This can only occur when $a=1/2$.

Therefore all zeros must have $a=1/2$.

IV. A. Summing Two Values of n in the $1/n^s$ Formula Using Two Opposing Spinners

We now look at summing $\frac{1}{n^s}$ for two or more integers of value n. For $n=2$ and $n=3$:

$$\frac{1}{2^s}+\frac{1}{3^s}\Big|_{(1-s)}^{s} = \left[\frac{1}{2^{(a+i\alpha)}}\right]-\left[\frac{1}{2^{(1-(a+i\alpha))}}\right] + \left[\frac{1}{3^{(a+i\alpha)}}\right]-\left[\frac{1}{3^{(1-(a+i\alpha))}}\right]$$

$$= \left[\left[\frac{1}{2^a}\cdot\frac{1}{2^{+i\alpha}}\right]-\left[\frac{1}{2^{(1-a)}}\cdot\frac{1}{2^{-i\alpha}}\right]\right] + \left[\left[\frac{1}{3^a}\cdot\frac{1}{3^{+i\alpha}}\right]-\left[\frac{1}{3^{(1-a)}}\cdot\frac{1}{3^{-i\alpha}}\right]\right] \qquad (6.14)$$

Here the components have been separated into an amplitude factor (a real number) times a complex number of

magnitude 1 (a unit spinner) as shown earlier. The unit spinners are described by $\frac{1}{n^{+i\alpha}}$ and $\frac{1}{n^{-i\alpha}}$, (and with exception of $n=1$), these are in general complex and irrational and of a different logarithmic base for each n, and are of magnitude 1. The unit spinners are multiplied by one of the real but irrational amplitude factors $\frac{1}{2^a}$, $\frac{1}{2^{(1-a)}}$, $\frac{1}{3^a}$, or $\frac{1}{3^{(1-a)}}$. The unit spinners have real and imaginary components that are irrational and of different logarithmic base for each n. In Figure 70 we plot $\left[\frac{1}{2^{(a+i\alpha)}}\right]-\left[\frac{1}{2^{(1-(a+i\alpha))}}\right]$, $\left[\frac{1}{3^{(a+i\alpha)}}\right]-\left[\frac{1}{3^{(1-(a+i\alpha))}}\right]$, and the sum of these two terms. This can also be written as $\frac{1}{2^s}\Big|_{(1-s)}^{s}$, $\frac{1}{3^s}\Big|_{(1-s)}^{s}$, and $\frac{1}{2^s}+\frac{1}{3^s}\Big|_{(1-s)}^{s}$ respectively. A value of $a=1/2$ was used.

$$F_2(\alpha) := \left[\frac{1}{2^{a+i\alpha}} - \frac{1}{2^{1-(a+i\alpha)}}\right] \qquad F_3(\alpha) := \left[\frac{1}{3^{a+i\alpha}} - \frac{1}{3^{1-(a+i\alpha)}}\right] \qquad F_\Sigma(\alpha) := F_2(\alpha) + F_3(\alpha)$$

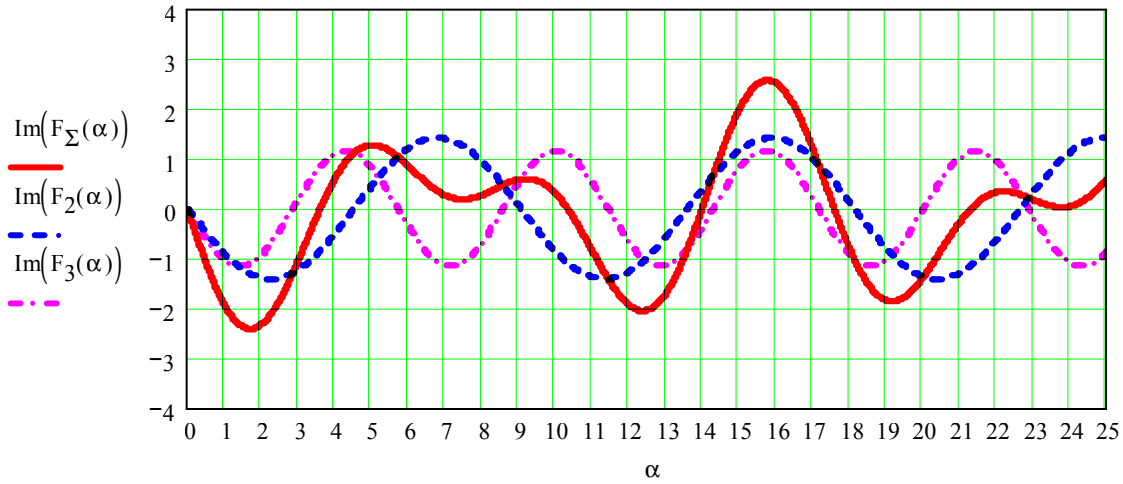


Figure 70 – The imaginary parts of $\frac{1}{2^s}\Big|_{(1-s)}^{s}$, $\frac{1}{3^s}\Big|_{(1-s)}^{s}$, and $\frac{1}{2^s}+\frac{1}{3^s}\Big|_{(1-s)}^{s}$ with $a=1/2$. The two imaginary component waves are of different frequencies which in general creates zeros. These zeros occur at irrational values of α that are specific to the component wave frequencies and amplitude factors. The real part is zero for all three expressions.

B. *a* equals 1/2

We will first work with $a = 1/2$. In order for $\frac{1}{2^s} + \frac{1}{3^s}\Big|^s_{(1-s)}$ to equal zero both the real and imaginary parts of $\frac{1}{2^s}\Big|^s_{(1-s)}$ and $\frac{1}{3^s}\Big|^s_{(1-s)}$ must sum to zero. We will let α_Σ denote the values of α that result in the imaginary part of $\frac{1}{2^s} + \frac{1}{3^s}\Big|^s_{(1-s)}$ to be zero.

This occurs when the imaginary part of $\left[\frac{1}{2^{1/2}} \cdot \frac{1}{2^{+i\alpha_\Sigma}}\right] - \left[\frac{1}{2^{1/2}} \cdot \frac{1}{2^{-i\alpha_\Sigma}}\right]$ equals the imaginary part of $-\left[\frac{1}{3^{1/2}} \cdot \frac{1}{3^{+i\alpha_\Sigma}}\right] + \left[\frac{1}{3^{1/2}} \cdot \frac{1}{3^{-i\alpha_\Sigma}}\right]$ and occurs at $\alpha_\Sigma = 3.60763...$ (irrational, rounded)

Now when $a = 1/2$ the real part of $\frac{1}{2^s} + \frac{1}{3^s}\Big|^s_{(1-s)}$ = $\left[\frac{1}{2^{1/2}} \cdot \frac{1}{2^{+i\alpha_\Sigma}}\right] - \left[\frac{1}{2^{1/2}} \cdot \frac{1}{2^{-i\alpha_\Sigma}}\right]$ subtracts out to zero as shown in #3 of the numerical example below. This is also true for $\frac{1}{3^s}\Big|^s_{(1-s)}$, and all $\frac{1}{n^s}\Big|^s_{(1-s)}$ in general. The real component of each spinner pair always subtracts to zero when $a = 1/2$.

So at $a = 1/2$ the real part of $\frac{1}{2^s}\Big|^s_{(1-s)}$ and $\frac{1}{3^s}\Big|^s_{(1-s)}$ are both zero. And at $\alpha = \alpha_\Sigma$ the imaginary part of $\frac{1}{2^s} + \frac{1}{3^s}\Big|^s_{(1-s)}$ will sum to zero.

This results in a zero for $\frac{1}{2^s} + \frac{1}{3^s}\Big|^s_{(1-s)}$ as shown in the numerical example below. The spinner model of this example is included as Figure 71.

1. First we find the individual terms:

$\alpha_\Sigma := 3.60763$

$\frac{1}{\frac{1}{2^2}} = 0.707107$ $\quad \frac{1}{2^{i\cdot\alpha_\Sigma}} = -0.801514 - 0.597976i$ $\quad \frac{1}{2^{-i\cdot\alpha_\Sigma}} = -0.801514 + 0.597976i$ ***n = 2***

$\frac{1}{\frac{1}{3^2}} = 0.577350$ $\quad \frac{1}{3^{i\cdot\alpha_\Sigma}} = -0.680908 + 0.732369i$ $\quad \frac{1}{3^{-i\cdot\alpha_\Sigma}} = -0.680908 - 0.732369i$ ***n = 3***

2. Next the products:

$\left(\frac{1}{\frac{1}{2^2}}\right)\cdot\left(\frac{1}{2^{i\cdot\alpha_\Sigma}}\right) = -0.566756 - 0.422833i$ $\quad \left(\frac{1}{\frac{1}{2^2}}\right)\cdot\left(\frac{1}{2^{-i\cdot\alpha_\Sigma}}\right) = -0.566756 + 0.422833i$ ***n = 2***

$\left(\frac{1}{\frac{1}{3^2}}\right)\cdot\left(\frac{1}{3^{i\cdot\alpha_\Sigma}}\right) = -0.393123 + 0.422833i$ $\quad \left(\frac{1}{\frac{1}{3^2}}\right)\cdot\left(\frac{1}{3^{-i\cdot\alpha_\Sigma}}\right) = -0.393123 - 0.422833i$ ***n = 3***

3. The real part of the products subtract out to zero for each *n*:

$\left(\frac{1}{\frac{1}{2^2}}\right)\cdot\left(\frac{1}{2^{i\cdot\alpha_\Sigma}}\right) - \left(\frac{1}{\frac{1}{2^2}}\right)\cdot\left(\frac{1}{2^{-i\cdot\alpha_\Sigma}}\right) = -0.845666i$ ***n = 2***

$\left(\frac{1}{\frac{1}{3^2}}\right)\cdot\left(\frac{1}{3^{i\cdot\alpha_\Sigma}}\right) - \left(\frac{1}{\frac{1}{3^2}}\right)\cdot\left(\frac{1}{3^{-i\cdot\alpha_\Sigma}}\right) = 0.845666i$ ***n = 3***

4. And the imaginary parts sum to zero over *n*:

$$\left[\left(\frac{1}{\frac{1}{2^2}}\right)\cdot\left(\frac{1}{2^{i\cdot\alpha_\Sigma}}\right) - \left(\frac{1}{\frac{1}{2^2}}\right)\cdot\left(\frac{1}{2^{-i\cdot\alpha_\Sigma}}\right)\right] + \left[\left(\frac{1}{\frac{1}{3^2}}\right)\cdot\left(\frac{1}{3^{i\cdot\alpha_\Sigma}}\right) - \left(\frac{1}{\frac{1}{3^2}}\right)\cdot\left(\frac{1}{3^{-i\cdot\alpha_\Sigma}}\right)\right] = 0.000000$$

and we have found a zero.

In general, the real part of any $\frac{1}{n^s}\Big|_{(1-s)}^{s}$ for a single (not summed) integer *n* subtracts to zero only at $a = \frac{1}{2}$ as shown in Section II.B. This is because at $a = \frac{1}{2}$, $a = (1-a)$ and the amplitude factors are equal for any pair of opposing spinners: $\frac{1}{n^a} = \frac{1}{n^{(1-a)}}$ at $a = \frac{1}{2}$. This results in the same amplitude factor applied to both unit spinners $\left[\frac{1}{n^{+i\alpha}}\right]$ and $\left[\frac{1}{n^{-i\alpha}}\right]$. So with $a = \frac{1}{2}$, $\left[\frac{1}{n^a}\cdot\frac{1}{n^{+i\alpha}}\right]-\left[\frac{1}{n^{(1-a)}}\cdot\frac{1}{n^{-i\alpha}}\right]$ reduces to $\frac{1}{n^{1/2}}\cdot\left[\frac{1}{n^{+i\alpha}}-\frac{1}{n^{-i\alpha}}\right]$ which further reduces to $\frac{1}{n^{1/2}}\cdot\left[2\cdot\mathrm{Im}\left(\frac{1}{n^{+i\alpha}}\right)\right]$.

The result is that for $a = \frac{1}{2}$ the real part of any complex spinner pair subtracts out to zero and the imaginary part doubles as shown in the numerical example above. When there are two values to sum, as in the case of $\frac{1}{2^s}+\frac{1}{3^s}\Big|_{(1-s)}^{s}$, the real parts of each spinner pair will subtract out to zero at $a = \frac{1}{2}$ leaving only an imaginary number for each spinner pair, $\frac{1}{2^{1/2}}\cdot\left[2\cdot\mathrm{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right]$ and $\frac{1}{3^{1/2}}\cdot\left[2\cdot\mathrm{Im}\left(\frac{1}{3^{+i\alpha}}\right)\right]$.

This expression corresponds to summing waves of different frequencies and their sum will have zeros as we see in Figure 70. Note that the real part is zero for all α due to the real part subtracting out for each spinner pair.

So the imaginary part of $\frac{1}{2^s}+\frac{1}{3^s}\Big|_{(1-s)}^{s}$ equals zero when these imaginary valued waves add to zero over *n*, and the real part equals zero because the real component of each spinner pair subtracts out to zero at $a = \frac{1}{2}$ for each *n*. This results in a zero for $\frac{1}{2^s}+\frac{1}{3^s}\Big|_{(1-s)}^{s}$.

In Figure 71 the spinner model of this same procedure is presented, and in Figure 72A it is shown that the relation can be modeled as two triangles which share a common side.

To summarize, the value of α which results in the imaginary part of $\frac{1}{2^s}+\frac{1}{3^s}\Big|_{(1-s)}^{s}$ to zero is called α_Σ. This corresponds to $\frac{1}{2^{1/2}}\cdot\left[2\cdot\mathrm{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right]$ and $\frac{1}{3^{1/2}}\cdot\left[2\cdot\mathrm{Im}\left(\frac{1}{3^{+i\alpha}}\right)\right]$ cancelling out to zero when summed. Note that $\frac{1}{2^{1/2}}$ and $\frac{1}{3^{1/2}}$ are the amplitude factors and are irrational and of different bases. Therefore α_Σ is highly specific

since it results in $\left[2\cdot \mathrm{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right]$ being a multiple of $2^{1/2}$ and $-\left[2\cdot \mathrm{Im}\left(\frac{1}{3^{+i\alpha}}\right)\right]$ being that same multiple of $3^{1/2}$. Thus the α_Σ value is like a key which exactly mates with the irrational amplitude factors and the imaginary part of the spinner positions. It moves the spinner pairs to the exact spot where the *imaginary* part of each spinner pair factors with the irrational amplitude factors yielding identical but negated products which sum to zero.

α_Σ corresponds to a different amount of rotation for each spinner pair. It can be converted to angle of rotation for the n^{th} pair using the relation $\pi - \frac{\alpha_\Sigma}{\alpha_n}\pi$. And $\alpha_n = i\cdot \log_{\frac{1}{n}}(-1)$ is the value corresponding to the *n*th spinner pair completing one half cycle. So α_Σ has the irrational amplitude factors and spinner positions for $n=2$ and $n=3$ encoded in it necessary for the imaginary part of $\frac{1}{2^s}+\frac{1}{3^s}\Big|_{(1-s)}^{s}$ to equal zero when $\alpha=\alpha_\Sigma$. And with $a = \frac{1}{2}$ the real part is also zero and therefore $\frac{1}{2^s}+\frac{1}{3^s}\Big|_{(1-s)}^{s}$ equals zero at these values.

α_Σ is specific to these four irrational values, the two amplitude factors and the two spinner components. In Figure 71 the spinner model is used to illustrate this. Figure 72A is a geometric model of the same using two right triangles which share a common side. The hypotenuse of each triangle is the amplitude factor. One leg represents the real component and one the imaginary component. The two imaginary components must be negatives of each other for the sum to be zero. Thus their absolute values must be equal, and so the two triangles share this common side. Figure 72B relates the initial wave model of Figure 70 to the two models presented in Figure 71 & 72A and shows that all three models are in fact equal.

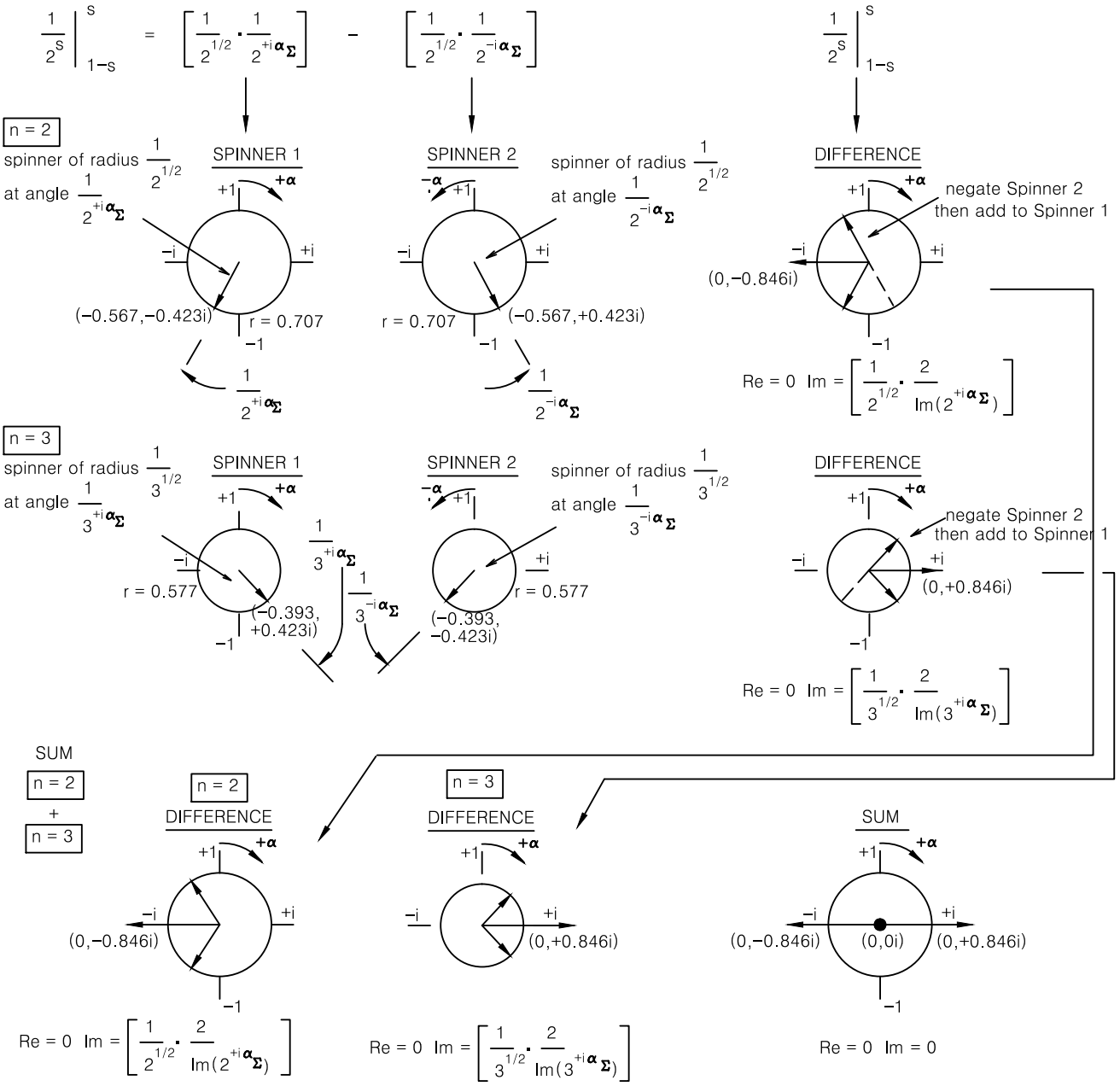
MODEL USING TWO OPPOSING SPINNERS WITH a = 1/2

In order for for $\left.\frac{1}{2^s}+\frac{1}{3^s}\right|_{1-s}^{s}$ to equal zero both its real and imaginary parts must sum to zero.

When a = 1/2 the real part of $\left.\frac{1}{2^s}\right|_{1-s}^{s}$ will subtract out to zero and the imaginary part will double.

$\left.\frac{1}{2^s}+\frac{1}{3^s}\right|_{1-s}^{s}$ can be modelled as two spinner pairs, one for each n, with the upper and lower evaluation boundaries determining the positions of each spinner in the pair. $\left.\frac{1}{n^s}\right|_{1-s}^{s}$ is the difference between the two spinners of a specific pair. The individual $\left.\frac{1}{n^s}\right|_{1-s}^{s}$ are then summed. When a = 1/2 the real part of each spinner pair subtracts out to zero.

Only the imaginary component remains. At $\alpha_\Sigma = 3.60763$ these imaginary components fromthe difference within the pairs also adds to zero.

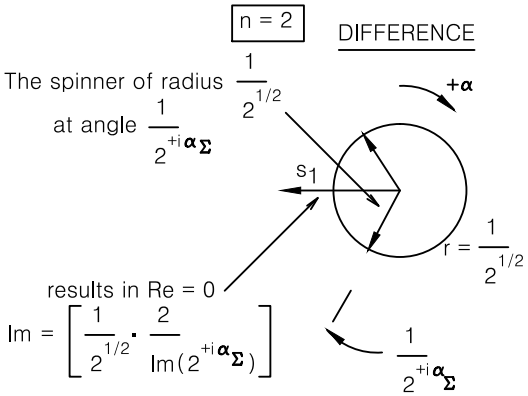


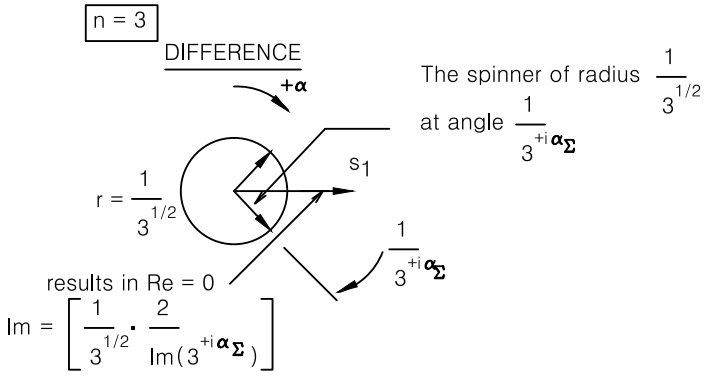
So for a = 1/2, the real component subtracts out to zero for each spinner pair, and the imaginary part doubles.
The value of α_Σ is such that the imaginary part also goes to zero when summed over n.

FIGURE 71

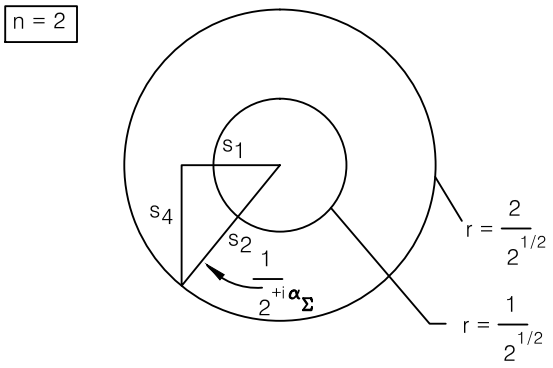
GEOMETRIC MODEL OF SUM OF SPINNER PAIRS n = 2 AND n = 3 WITH a = 1/2

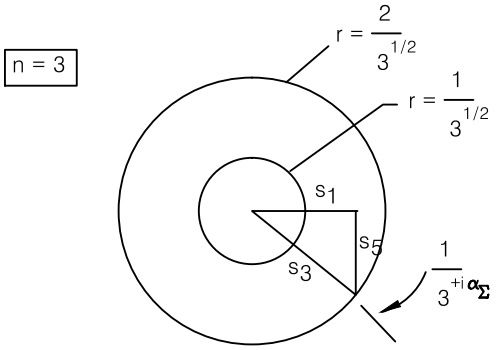
Evaluating $\frac{1}{2^s}\Big|_{1-s}^{s}$ the spinner of radius $\frac{1}{2^{1/2}}$ at position $\frac{1}{2^{+i\alpha_\Sigma}}$ results in the real component subtracting to zero and the imaginary component doubling to $\frac{1}{2^{1/2}} \cdot \frac{2}{\mathrm{Im}(2^{+i\alpha_\Sigma})}$. A similar relationship exists for n = 3.





When α is chosen so that these resultants are equal but opposite they will sum to zero. This value of α is α_Σ .
We can use this to create a geometric relationship with two triangles sharing a similar side.
At α_Σ the side s_1 must have the same length for the imaginary parts to sum to zero.





The n = 3 triangle can be mirrored and aligined with the n = 2 triangle so that s_1 is common to both.
The circles are doubled to match the hypotenuse sides.

$$s_1 = \left[\frac{1}{2^{1/2}} \cdot \frac{2}{\mathrm{Im}(2^{+i\alpha_\Sigma})}\right] = \left[\frac{1}{3^{1/2}} \cdot \frac{2}{\mathrm{Im}(2^{+i\alpha_\Sigma})}\right]$$

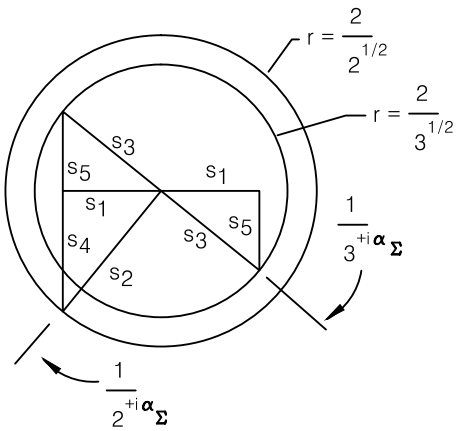
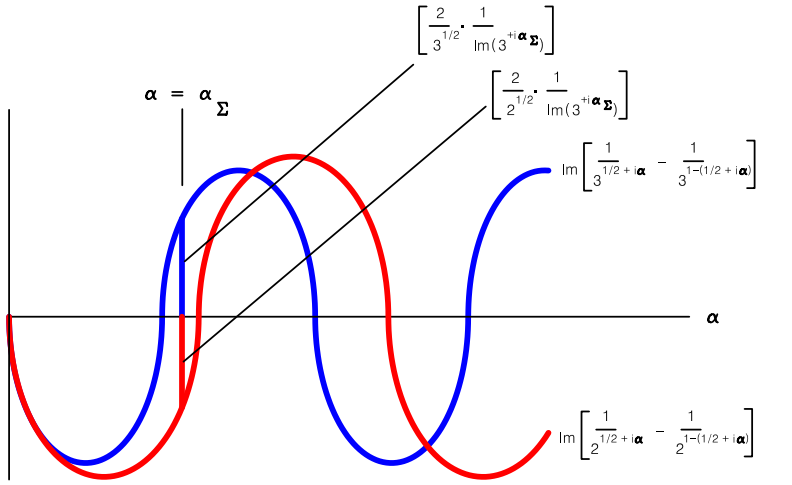


FIGURE 72A

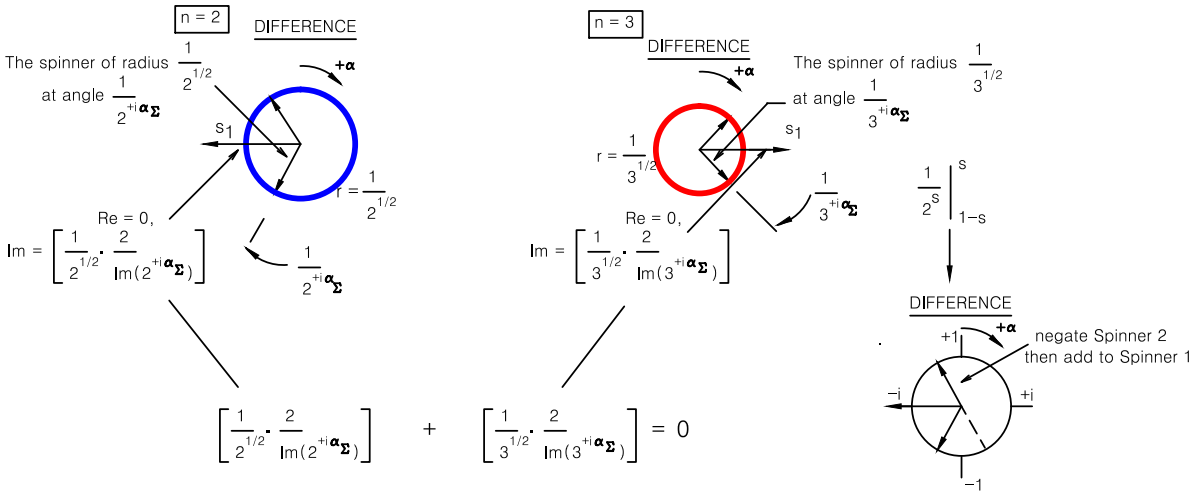
WAVE MODEL

The imaginary component of the wave equations for n = 2 and n = 3 are graphed for a = 1/2.
At $\alpha = \alpha_\Sigma$ the imaginary components are equal but opposite in sign and sum to zero.



SPINNER MODEL

The real component of each spinner subtracts out to zero and the imaginary components double.
At α_Σ the two resultants are of exact magnitude to negate to zero.



GEOMETRIC MODEL

The radius of each circle represents the magnitude of each wavefunction and determines the length of the hypotenuse of a right triangle. One leg is the imaginary component and the other is the real one. The slope of each hypotenuse is determined by α. At α_Σ the imaginary components are equal but opposite in sign resulting in the two triangles having a common side.

At α_Σ $\left[\frac{-2}{3^{1/2}} \cdot \frac{1}{\mathrm{Im}(3^{+i\alpha_\Sigma})}\right] = \left[\frac{+2}{2^{1/2}} \cdot \frac{1}{\mathrm{Im}(2^{+i\alpha_\Sigma})}\right]$

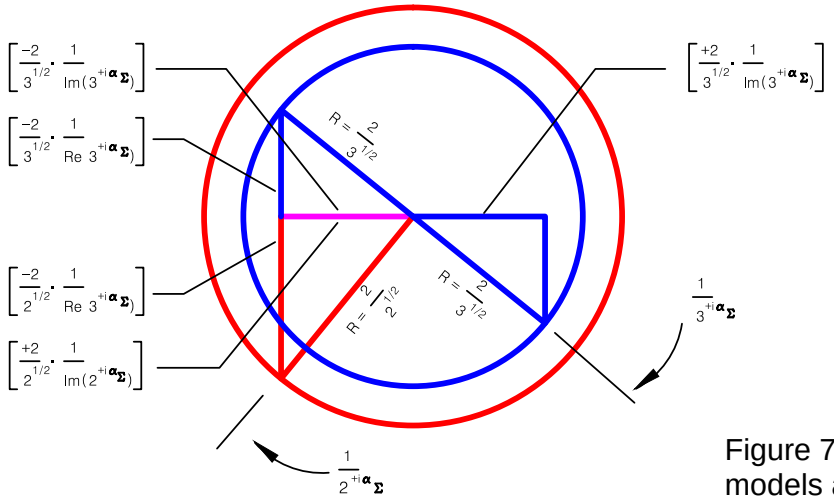


Figure 72B - The three models are equivalent.

C. *a* not equal to 1/2

The previous evaluation was for $a = 1/2$. That is the only way for $\frac{1}{2^s} + \frac{1}{3^s}\Big|_{(1-s)}^{s}$ to have a value of zero as we will show. Suppose now $a \neq 1/2$. If we let $a = 1/4$ for example, then for the spinner pairs $\frac{1}{2^s}\Big|_{(1-s)}^{s}$ and $\frac{1}{3^s}\Big|_{(1-s)}^{s}$ we can still find a value for α_Σ that causes the ***imaginary*** equation to zero:

$\alpha_\Sigma := 3.59622$

$$\left(\frac{1}{\frac{1}{2^4}}\right)\cdot\left(\frac{1}{2^{i\cdot\alpha_\Sigma}}\right) - \left(\frac{1}{\frac{3}{2^4}}\right)\cdot\left(\frac{1}{2^{-i\cdot\alpha_\Sigma}}\right) = -0.196236 - 0.867468i \qquad \textbf{n = 2}$$

$$\left(\frac{1}{\frac{1}{3^4}}\right)\cdot\left(\frac{1}{3^{i\cdot\alpha_\Sigma}}\right) - \left(\frac{1}{\frac{3}{3^4}}\right)\cdot\left(\frac{1}{3^{-i\cdot\alpha_\Sigma}}\right) = -0.221601 + 0.867465i \qquad \textbf{n = 3}$$

And the sum of these is

$$\left[\left(\frac{1}{\frac{1}{2^4}}\right)\cdot\left(\frac{1}{2^{i\cdot\alpha_\Sigma}}\right) - \left(\frac{1}{\frac{3}{2^4}}\right)\cdot\left(\frac{1}{2^{-i\cdot\alpha_\Sigma}}\right)\right] + \left[\left(\frac{1}{\frac{1}{3^4}}\right)\cdot\left(\frac{1}{3^{i\cdot\alpha_\Sigma}}\right) - \left(\frac{1}{\frac{3}{3^4}}\right)\cdot\left(\frac{1}{3^{-i\cdot\alpha_\Sigma}}\right)\right] = -0.417837$$

Thus with $a \neq 1/2$ we are still able to find values of α that cause the ***imaginary part*** of $\frac{1}{2^s} + \frac{1}{3^s}\Big|_{(1-s)}^{s}$ to equal zero since adding waves of different frequencies will in general have zeros. See Figure 73A. ***But the real part is no longer zero everywhere***. See Figure 73B. It too is wave based , and it will never sum to zero at the same value of α that the imaginary part does as will be shown. And in order for the overall expression to have a zero both the real and imaginary parts must sum to zero at the same value of α and that cannot happen with $a \neq 1/2$.

In Figure 73B we plot the imaginary and real values of $\frac{1}{2^s} + \frac{1}{3^s}\Big|_{(1-s)}^{s}$ with $a = 1/4$. Note that the real components are no longer always zero. These are now a summation of waves with their own set of irrational α values that result in zeros. However the α values that zero the imaginary part do not correspond with the α values that zero the real part. If we examine Figures 73C & 73D we see that the imaginary and real parts of both $\frac{1}{2^s}\Big|_{(1-s)}^{s}$ and $\frac{1}{3^s}\Big|_{(1-s)}^{s}$ are out of phase. When one part is zero, the other is at a maxima or minima. So it is not that surprising that for the sum $\frac{1}{2^s} + \frac{1}{3^s}\Big|_{(1-s)}^{s}$ any α value that cause the imaginary part to zero cannot result in a zero for the real part.

$$F_2(\alpha) := \left[\frac{1}{2^{a+i\alpha}} - \frac{1}{2^{1-(a+i\alpha)}} \right] \qquad F_3(\alpha) := \left[\frac{1}{3^{a+i\alpha}} - \frac{1}{3^{1-(a+i\alpha)}} \right] \qquad F_\Sigma(\alpha) := F_2(\alpha) + F_3(\alpha)$$

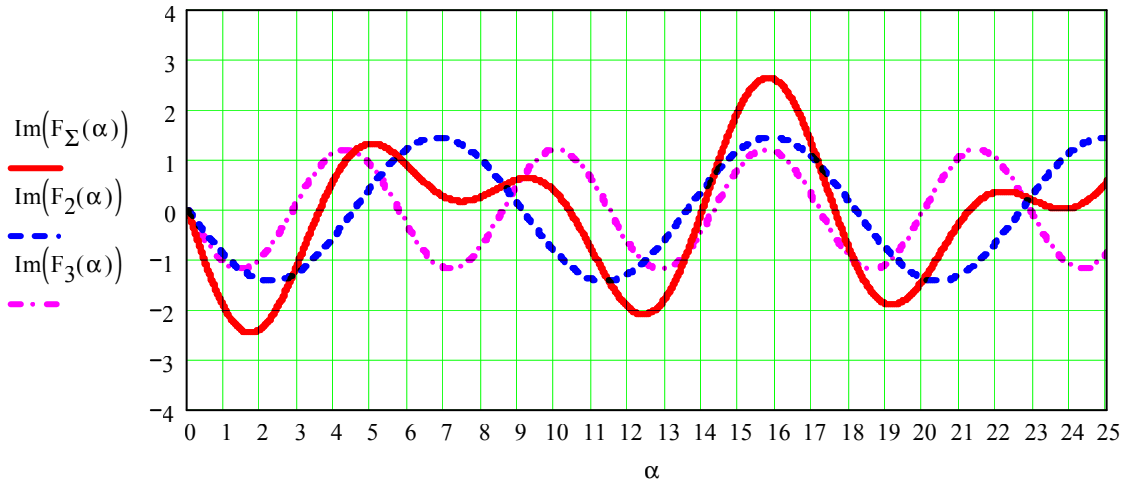


Figure 73A – The imaginary part of $\frac{1}{2^s} + \frac{1}{3^s}\Big|^s_{(1-s)}$, $\frac{1}{2^s}\Big|^s_{(1-s)}$, and $\frac{1}{2^s}\Big|^s_{(1-s)}$ with $a = 1/4$. The component waves have different frequencies which produce imaginary zeros when summed.

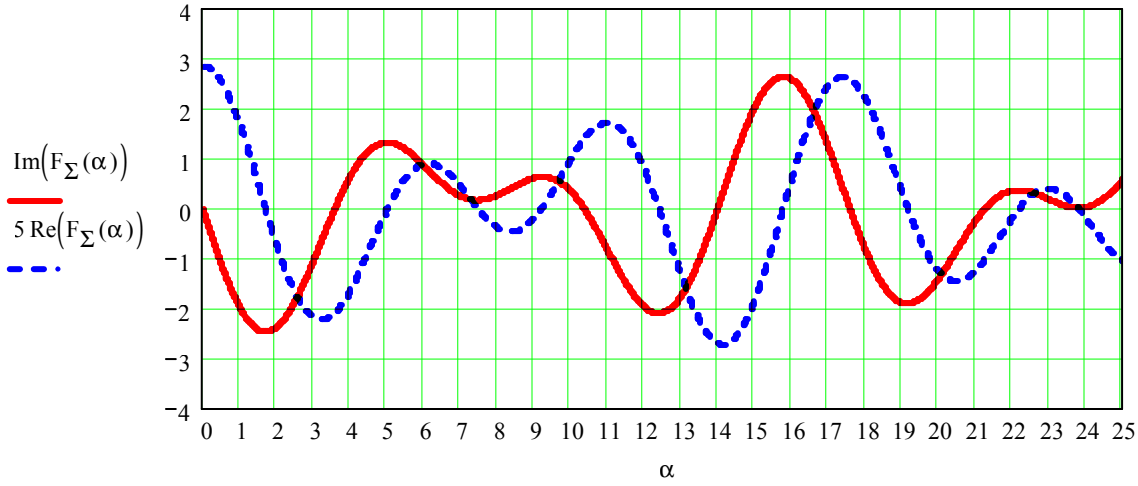
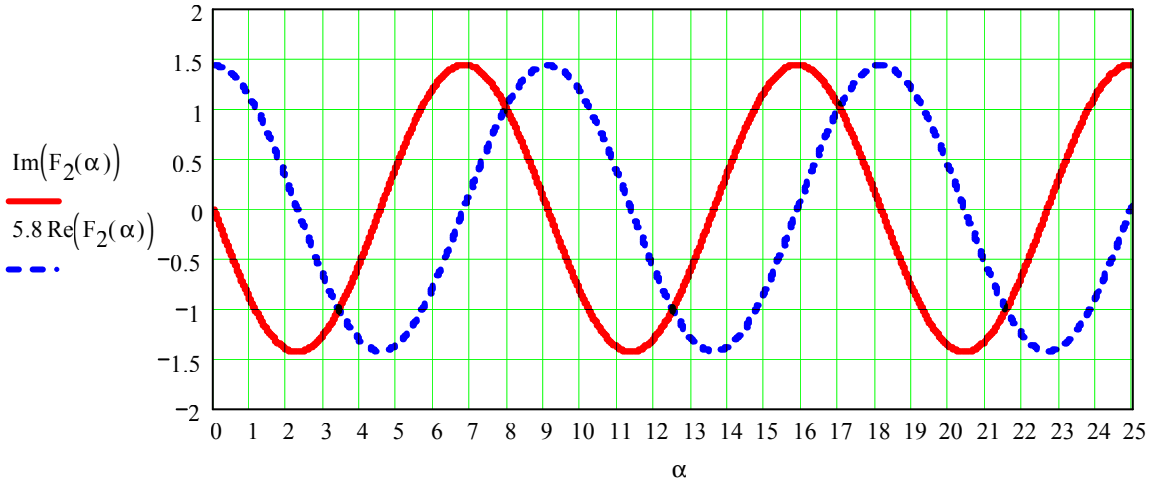


Figure 73B – The real and imaginary parts of $\frac{1}{2^s} + \frac{1}{3^s}\Big|^s_{(1-s)}$ with $a = 1/4$. The real component has been magnified vertically and is no longer zero for all α. It has its own set of irrational α values that result in zeros.



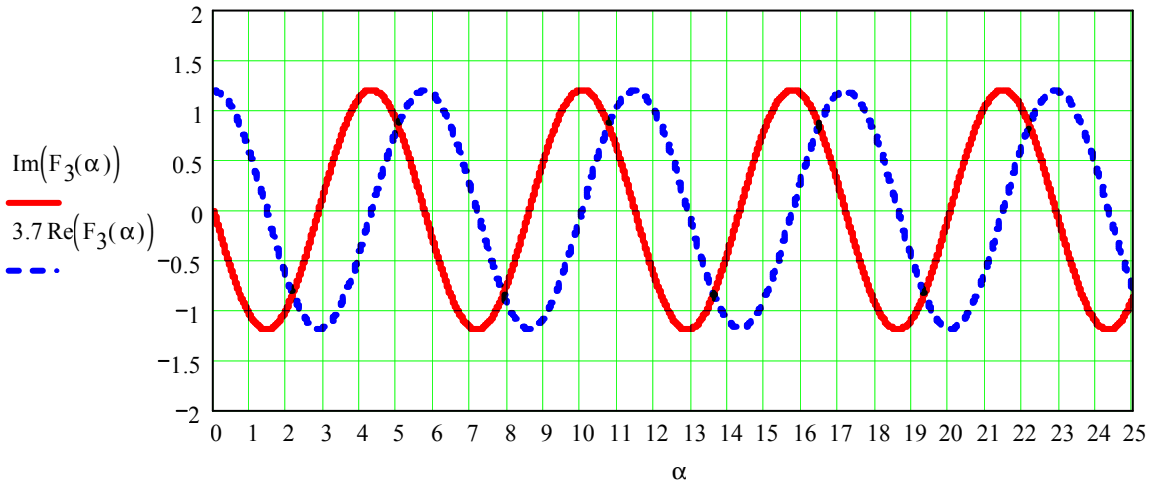


Figure 73C & D – The real and imaginary parts of $\frac{1}{2^s}\Big|_{(1-s)}^{s}$ (top) and $\frac{1}{3^s}\Big|_{(1-s)}^{s}$ (bottom) with $a = 1/4$. The real components have been magnified vertically. Note the real and imaginary components are out of phase by one quarter cycle. When one is zero, the other is at a maxima or minima.

So when $a \neq 1/2$, $\frac{1}{n^s}\Big|_{(1-s)}^{s}$ does not reduce to $\frac{1}{n^{1/2}} \cdot \left[\frac{2}{n^{+i\cdot\alpha}}\right]$ and the real part of each spinner pair does not subtract out to zero. So for $n = 2$ and $n = 3$:

$$\frac{1}{2^s}+\frac{1}{3^s}\Big|_{(1-s)}^{s} = \left[\frac{1}{2^{(a+i\cdot\alpha)}}\right]-\left[\frac{1}{2^{(1-(a+i\cdot\alpha))}}\right] + \left[\frac{1}{3^{(a+i\cdot\alpha)}}\right]-\left[\frac{1}{3^{(1-(a+i\cdot\alpha))}}\right]$$

$$= \left[\left[\frac{1}{2^a}\cdot\frac{1}{2^{+i\cdot\alpha}}\right]-\left[\frac{1}{2^{(1-a)}}\cdot\frac{1}{2^{-i\cdot\alpha}}\right]\right] + \left[\left[\frac{1}{3^a}\cdot\frac{1}{3^{+i\cdot\alpha}}\right]-\left[\frac{1}{3^{(1-a)}}\cdot\frac{1}{3^{-i\cdot\alpha}}\right]\right] \qquad (6.19)$$

If we separate the imaginary and real components, the imaginary part of $\frac{1}{2^s}+\frac{1}{3^s}\Big|_{(1-s)}^{s}$ with $a = 1/4$ is

$$\left[\frac{1}{2^{\frac{1}{4}}}\right]\cdot\left[\mathrm{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right]-\left[\frac{1}{2^{\frac{3}{4}}}\right]\cdot\left[\mathrm{Im}\left(\frac{1}{2^{-i\alpha}}\right)\right] \;+\; \left[\frac{1}{3^{\frac{1}{4}}}\right]\cdot\left[\mathrm{Im}\left(\frac{1}{3^{+i\alpha}}\right)\right]-\left[\frac{1}{3^{\frac{3}{4}}}\right]\cdot\left[\mathrm{Im}\left(\frac{1}{3^{-i\alpha}}\right)\right] \qquad (6.20A)$$

and the real part is

$$\left[\frac{1}{2^{\frac{1}{4}}}\right]\cdot\left[\mathrm{Re}\left(\frac{1}{2^{+i\alpha}}\right)\right]-\left[\frac{1}{2^{\frac{3}{4}}}\right]\cdot\left[\mathrm{Re}\left(\frac{1}{2^{-i\alpha}}\right)\right] \;+\; \left[\frac{1}{3^{\frac{1}{4}}}\right]\cdot\left[\mathrm{Re}\left(\frac{1}{3^{+i\alpha}}\right)\right]-\left[\frac{1}{3^{\frac{3}{4}}}\right]\cdot\left[\mathrm{Re}\left(\frac{1}{3^{+i\alpha}}\right)\right] \qquad (6.21A)$$

The relations $\left[\mathrm{Im}\left(\frac{1}{2^{-i\alpha}}\right)\right] = -\left[\mathrm{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right]$ and $\left[\mathrm{Re}\left(\frac{1}{2^{-i\alpha}}\right)\right] = -\left[\mathrm{Re}\left(\frac{1}{2^{+i\alpha}}\right)\right]$ are apparent from the unit circle. Substituting we obtain two equations, one imaginary and one real. We need *both* equations to equal zero for $\frac{1}{2^s}+\frac{1}{3^s}\Big|_{(1-s)}^{s}$ to equal zero. We will set the imaginary equation equal to zero and see if it is permissible for the real equation to equal zero simultaneously:

[Im] If $$\left[\frac{1}{2^{\frac{1}{4}}}+\frac{1}{2^{\frac{3}{4}}}\right]\cdot\left[\mathrm{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right] \;+\; \left[\frac{1}{3^{\frac{1}{4}}}+\frac{1}{3^{\frac{3}{4}}}\right]\cdot\left[\mathrm{Im}\left(\frac{1}{3^{+i\alpha}}\right)\right] = 0 \qquad (6.20B)$$

[Re] Can $$\left[\frac{1}{2^{\frac{1}{4}}}-\frac{1}{2^{\frac{3}{4}}}\right]\cdot\left[\mathrm{Re}\left(\frac{1}{2^{+i\alpha}}\right)\right] \;+\; \left[\frac{1}{3^{\frac{1}{4}}}-\frac{1}{3^{\frac{3}{4}}}\right]\cdot\left[\mathrm{Re}\left(\frac{1}{3^{+i\alpha}}\right)\right] = 0? \qquad (6.21B)$$

We recognize that $\left[\mathrm{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right]$ and $\left[\mathrm{Re}\left(\frac{1}{2^{+i\alpha}}\right)\right]$ as the legs of a right triangle inscribed in a unit spinner and can be expressed in terms of sine and cosine of the angle β_2. See Figure 72C. A similar relation applies for the $n = 3$ components and angle β_3. (We recall that β_2 and β_3 are also both functions of α.) If we split the amplitude factors into the components $A_2 = \frac{1}{2^{\frac{1}{4}}}$, $B_2 = \frac{1}{2^{\frac{3}{4}}}$, $A_3 = \frac{1}{3^{\frac{1}{4}}}$, and $B_3 = \frac{1}{3^{\frac{3}{4}}}$. We then can see (6.20B) & (6.21B) are of the form:

[Im] If $$[A_2 + B_2]\cdot[\sin(\beta_2)] \;+\; [A_3 + B_3]\cdot[\sin(\beta_3)] = 0 \qquad (6.20C)$$

[Re] Can $$[A_2 - B_2]\cdot[\cos(\beta_2)] \;+\; [A_3 - B_3]\cdot[\cos(\beta_3)] = 0? \qquad (6.21C)$$

Can the values of β_2 and β_3 that result in zero for (6.20C) also result in a zero for (6.21C)? In (6.21C) the value of $[\sin(\beta_2)]$ is specifically matched to multiply with the amplitude factor $[A_2 + B_2]$, and this product negates with another exact match, the product of $[\sin(\beta_3)]$ and $[A_3 + B_3]$. Can we expect the cosines of the same angles to match with $[A_2 - B_2]$ and $[A_3 - B_3]$ and negate in the same fashion?

No. The sine of each β_n angle is irrational and matches multiplying with an irrational amplitude factor $[A_n + B_n]$ in (6.20C). When the cosines of these same β_n angles (in 6.21C) try to match up multiplying with a linear combination of the same irrational amplitudes A_n and B_n and sum to zero, this is not permissible. That is because the cosine is also irrational and is of a different base than the sine: $\cos(\beta_n) = \pm\sqrt{1-\sin^2(\beta_n)}$. Thus the cosine involves the square root of a different number than the sine. So if $[\sin(\beta_2)]$ and $[\sin(\beta_3)]$ zero with $[A_2 + B_2]$, and $[A_3 + B_3]$, respectively, then $[\cos(\beta_2)]$ and $[\cos(\beta_3)]$ cannot zero with any (rational) linear combinations of A_n and B_n. Therefore:

[Im] If $[A_2 + B_2]\cdot[\sin(\beta_2)] + [A_3 + B_3]\cdot[\sin(\beta_3)] = 0$ (6.20D)

[Re] Then $[a_2 A_2 + b_2 B_2]\cdot[\cos(\beta_2)] + [a_3 A_3 + b_3 B_3]\cdot[\cos(\beta_3)] \neq 0$ (6.21D)

where the a_n and b_n are non-zero integers or integer ratios (rational numbers). Note that (6.20C/6.21C) is a particular case of the more general equation (6.20D/6.21D) with a_2 & $a_3 = 1$ and b_2 & $b_3 = -1$. So $[A_n - B_n]$ is in fact a rational linear combination of $[A_n + B_n]$. If (6.20C) zeros then (6.21C) cannot.

[Im] If $[A_2 + B_2]\cdot[\sin(\beta_2)] + [A_3 + B_3]\cdot[\sin(\beta_3)] = 0$ (6.20E)

[Re] Then $[A_2 - B_2]\cdot[\cos(\beta_2)] + [A_3 - B_3]\cdot[\cos(\beta_3)] \neq 0$ (6.21E)

The argument is more easily understood with an analogy using square roots. Consider this example:

[Im] If $[A_2 + B_2]\cdot[\sqrt{2}] + [A_3 + B_3]\cdot[\sqrt{3}] = 0$ (6.20F)

[Re] Then $[a_2 A_2 + b_2 B_2]\cdot[\sqrt{7}] + [a_3 A_3 + b_3 B_3]\cdot[\sqrt{13}] \neq 0$ (6.21F)

In other words, anything built out of summing $\sqrt{2}$'s and $\sqrt{3}$'s, like a zero, cannot be made from any rational linear combination of $\sqrt{7}$'s and $\sqrt{13}$'s. And by extension, anything created from summing $A_2\sqrt{2}$, $B_2\sqrt{2}$, $A_3\sqrt{3}$, and $B_3\sqrt{3}$ cannot be made from any rational linear combination of $A_2\sqrt{7}$, $B_2\sqrt{7}$, $A_3\sqrt{13}$, and $B_3\sqrt{13}$. In (6.20F) the constants A_2 and B_2 must mate with $\sqrt{2}$ to zero. Therefore a rational linear

combination of these same constants cannot mate with $\sqrt{7}$ in (6.21F) because $\sqrt{7}$ is a different irrational base. Its signature of digits will never mate with products that sum to zero where $\sqrt{2}$ did.

(6.20F/6.21F) is an analogy to illustrate the mechanics we expect in (6.20D/6.21D), (6.20C/6.21C), and (6.20B/6.21B). There is no numerical equivalence between these. The point is to show that the sine and cosine (or real and imaginary components) of a common angle are the ratios of the square root of different base numbers and the same type of relation must hold. Whatever we obtain through a linear combination of sine functions cannot be obtained through a linear combination of cosine functions of the same angles. One is tempted to think that there is an angle such as $\pi/4$ or 45^{o} where both imaginary and real components are equal and would zero simultaneously. However the α values are irrational and no exact value such as $\pi/4$ is permissible.

In general, numbers of different square root bases do not add together, they must be kept algebraically separate. Each is irrational and contains an infinite number of unique non-repeating digits. These can never combine through addition to produce any exact number including zero. The only time irrational numbers combine to create rational numbers are 1) when added or subtracted to a multiple of its own base to result in zero, 2) when divided by itself or multiplied by its reciprocal or a multiple of it, and in this way cancelling out the irrational base, or 3) through inverse operation such as exponent, logarithm, or squaring.

Summary of Section IV

The previous section is for $a \neq 1/2$. The numerical examples used $a = 1/4$ as an illustration but the results are applicable to all values where $a \neq 1/2$. α is defined continuously and the imaginary component of $\frac{1}{2^s}+\frac{1}{3^s}\Big|_{(1-s)}^{s}$ crosses zero as seen in Figure 73A. (6.20B) and (6.21B) are the imaginary and real equations that $\frac{1}{2^s}+\frac{1}{3^s}\Big|_{(1-s)}^{s}$ was separated into. Thus there are α values where (6.20B), which is the imaginary component, sums to zero.

Examining the imaginary equation we see that the α values that zero (6.20B) are highly specific. They result in the spinner pairs turning the exact amount so that the product of $\left[\frac{1}{2^{\frac{1}{4}}}+\frac{1}{2^{\frac{3}{4}}}\right]$ and $\left[\mathrm{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right]$ added to the product of $\left[\frac{1}{3^{\frac{1}{4}}}+\frac{1}{3^{\frac{3}{4}}}\right]$ and $\left[\mathrm{Im}\left(\frac{1}{3^{+i\alpha}}\right)\right]$ sum to zero. Those α values are a key specific to the zeroing of (6.20B). They encode the exact positions where the spinners match with the amplitude factors and result in imaginary components which sum to zero.

In order to zero (6.20B): $\left[\frac{1}{2^{\frac{1}{4}}}+\frac{1}{2^{\frac{3}{4}}}\right]\cdot\left[\mathrm{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right] + \left[\frac{1}{3^{\frac{1}{4}}}+\frac{1}{3^{\frac{3}{4}}}\right]\cdot\left[\mathrm{Im}\left(\frac{1}{3^{+i\alpha}}\right)\right] = 0$

a cancellation or "mating" must occur. $\left[\mathrm{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right]$ must contain the factor $\left[\frac{1}{2^{\frac{1}{4}}}+\frac{1}{2^{\frac{3}{4}}}\right]^{-1}$

and $\left[\mathrm{Im}\left(\frac{1}{3^{+i\alpha}}\right)\right]$ must contain the factor $\left[\frac{1}{3^{\frac{1}{4}}}+\frac{1}{3^{\frac{3}{4}}}\right]^{-1}$.

Using the factor K common to both, the cancellation will occur resulting in $K-K=0$ if

$$\left[\mathrm{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right] = \left[\frac{1}{2^{\frac{1}{4}}}+\frac{1}{2^{\frac{3}{4}}}\right]^{-1}\cdot\mathrm{K} \qquad \left[\mathrm{Im}\left(\frac{1}{3^{+i\alpha}}\right)\right] = -\left[\frac{1}{3^{\frac{1}{4}}}+\frac{1}{3^{\frac{3}{4}}}\right]^{-1}\cdot\mathrm{K} \tag{6.22}$$

which we can demonstrate by substituting (6.22) into (6.20B):

$$\left[\frac{1}{2^{\frac{1}{4}}}+\frac{1}{2^{\frac{3}{4}}}\right]\cdot\left[\left[\frac{1}{2^{\frac{1}{4}}}+\frac{1}{2^{\frac{3}{4}}}\right]^{-1}\cdot\mathrm{K}\right] + \left[\frac{1}{3^{\frac{1}{4}}}+\frac{1}{3^{\frac{3}{4}}}\right]\cdot\left[-\left[\frac{1}{3^{\frac{1}{4}}}+\frac{1}{3^{\frac{3}{4}}}\right]^{-1}\cdot\mathrm{K}\right] = 0 \tag{6.23}$$

And we do get $K-K=0$.

$\left[\mathrm{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right]$ and $\left[\mathrm{Re}\left(\frac{1}{2^{+i\alpha}}\right)\right]$ represent different legs of a right triangle inscribed on a unit circle. These coefficients are irrational are of different bases as we understand from their correspondence with sine and cosine. They are defined by the unit vector and unit circle and obey the formula $\left[\mathrm{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right]^2 + \left[\mathrm{Re}\left(\frac{1}{2^{+i\alpha}}\right)\right]^2 = 1$ which can be rewritten as $\left[\mathrm{Re}\left(\frac{1}{2^{+i\alpha}}\right)\right] = \pm\sqrt{1-\left[\mathrm{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right]^2}$. Therefore $\left[\mathrm{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right]$ and $\left[\mathrm{Re}\left(\frac{1}{2^{+i\alpha}}\right)\right]$ involve the square roots of different numbers and are irrational and of different bases.

(6.20B) and (6.21B) are the imaginary and real equations that $\frac{1}{2^s}+\frac{1}{3^s}\Big|_{(1-s)}^{s}$ was separated into. In these equations $\left[\mathrm{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right]$ and $\left[\mathrm{Re}\left(\frac{1}{2^{+i\alpha}}\right)\right]$ are both multiplied by a linear combination of the same underlying irrational amplitude factors $\frac{1}{2^{\frac{1}{4}}}$ and $\frac{1}{2^{\frac{3}{4}}}$. The same is true for the $n=3$ terms. Since the amplitude factors are linear combinations of the same underlying irrational values for both the imaginary and real component equations, the α value that results in a zero for the imaginary equation cannot result in a zero for the real equation.

In other words:

$$[\mathrm{Im}] \quad \text{If} \quad \left[\frac{1}{2^{\frac{1}{4}}}+\frac{1}{2^{\frac{3}{4}}}\right]\cdot\left[\mathrm{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right] + \left[\frac{1}{3^{\frac{1}{4}}}+\frac{1}{3^{\frac{3}{4}}}\right]\cdot\left[\mathrm{Im}\left(\frac{1}{3^{+i\alpha}}\right)\right] = 0 \qquad (6.20G)$$

$$[\mathrm{Re}] \quad \text{Then} \quad \left[\frac{1}{2^{\frac{1}{4}}}-\frac{1}{2^{\frac{3}{4}}}\right]\cdot\left[\frac{1}{\pm\sqrt{1-\left[\mathrm{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right]^2}}\right] + \left[\frac{1}{3^{\frac{1}{4}}}-\frac{1}{3^{\frac{3}{4}}}\right]\cdot\left[\frac{1}{\pm\sqrt{1-\left[\mathrm{Im}\left(\frac{1}{3^{+i\alpha}}\right)\right]^2}}\right] \neq 0 \qquad (6.21G)$$

or equivalently that they cannot both zero at the same α value. This implies that:

$$[\mathrm{Im}] \quad \text{If} \quad \left[\frac{1}{2^{\frac{1}{4}}}+\frac{1}{2^{\frac{3}{4}}}\right]\cdot\left[\mathrm{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right] + \left[\frac{1}{3^{\frac{1}{4}}}+\frac{1}{3^{\frac{3}{4}}}\right]\cdot\left[\mathrm{Im}\left(\frac{1}{3^{+i\alpha}}\right)\right] = 0 \qquad (6.20H)$$

$$[\mathrm{Re}] \quad \text{Then} \quad \left[\frac{1}{2^{\frac{1}{4}}}-\frac{1}{2^{\frac{3}{4}}}\right]\cdot\left[\mathrm{Re}\left(\frac{1}{2^{+i\alpha}}\right)\right] + \left[\frac{1}{3^{\frac{1}{4}}}-\frac{1}{3^{\frac{3}{4}}}\right]\cdot\left[\mathrm{Re}\left(\frac{1}{3^{+i\alpha}}\right)\right] \neq 0 \qquad (6.21H)$$

which answers our original question about (6.20B).

So in the case of $a \neq 1/2$ when α is chosen to yield an imaginary zero, the imaginary components multiply with the irrational amplitude factors and these products sum to yield zero. The value of the imaginary components match specifically to those amplitude factors mating with their irrational bases and the α value encodes this. The real components are of a different irrational base and can never zero with any rational linear combinations of these same amplitude factors at the same value of α that zeros the imaginary equation. In order for the real components

to multiply by a linear combination of these same irrational amplitude factors with products that sum to zero, the real components must also be specifically matched to mate with the amplitude factors in a similar way that was shown for the irrational equation. An α value corresponding to the real equation zeroing would irrationally encode this condition of the real components matching and mating, but this α value would no longer zero the imaginary component equation.

Thus it is not possible for both (6.20A-G) and (6.21A-G) to zero at a common α value when $a \neq 1/2$ If one of the component equations zeros then the other cannot.

Therefore $\frac{1}{2^s} + \frac{1}{3^s}\Big|_{(1-s)}^{s}$ has no zeros when $a \neq 1/2$ because the real and imaginary components cannot zero simultaneously as shown through separating the component equations. All zeros of $\frac{1}{2^s} + \frac{1}{3^s}\Big|_{(1-s)}^{s}$ must occur at $a = 1/2$. At this a value the real component of the two spinners in each pair subtracts out to zero and there are only imaginary components remaining. Then at the specific value $\alpha = \alpha_\Sigma$ the imaginary part of the two spinner pairs sums to zero resulting in a zero for the overall expression.

To express this mathematically we rewrite (6.20B/6.21B) in terms of the more general variable a:

[Im] With $\alpha = \alpha_\Sigma$ $$\left[\frac{1}{2^a} + \frac{1}{2^{1-a}}\right]\cdot\left[\mathrm{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right] + \left[\frac{1}{3^a} + \frac{1}{3^{1-a}}\right]\cdot\left[\mathrm{Im}\left(\frac{1}{3^{+i\alpha}}\right)\right] = 0 \qquad (6.20i)$$

[Re] With $a = 1/2$ $$\left[\frac{1}{2^a} - \frac{1}{2^{1-a}}\right]\cdot\left[\mathrm{Re}\left(\frac{1}{2^{+i\alpha}}\right)\right] + \left[\frac{1}{3^a} - \frac{1}{3^{1-a}}\right]\cdot\left[\mathrm{Re}\left(\frac{1}{3^{+i\alpha}}\right)\right] = 0 \qquad (6.21i)$$

So when $a = 1/2$ the amplitude factors $\left[\frac{1}{2^a} - \frac{1}{2^{1-a}}\right]$ and $\left[\frac{1}{3^a} - \frac{1}{3^{1-a}}\right]$ of the real equation (6.21i) both equal zero. That is how (6.20i) and (6.21i) zero simultaneously. This is equivalent to saying that the real part cancels out for each spinner pair. The imaginary part zeros at $\alpha = \alpha_\Sigma$ where the spinner positions are such that the imaginary component mate with the irrational amplitude factor in multiplication, and the sum of these products negates to zero.

Importance

The importance of this model is what it tells us about the general number system. We consider the integers *n* part of a larger number set n^s which includes the extra variables $n^{i\alpha}$ and n^a in addition to n.

A mapping transformation is required to appreciate the significance of n^s. A model with two opposing spinners is used to depict the mapping. The use of mapping transformations was introduced in the first paper. Some physical events exhibit better mapping characteristics in a transformed space and this may be the more logical frame to view them from. The GTE requires the results of the two maps to be graphically equivalent since they describe the results of a single event mapped in two different ways.

The α value defines the phase or position of the two spinners in the complex plane. At certain α values specific to *n*, the imaginary parts will sum to zero and there is no imaginary component. This set of α values $\alpha = \alpha_{nk}$ corresponds with the real numbers.

The a value defines the radius of each spinner pair. For the specific value a = ½ the real part of each spinner pair will sum to zero and there is no real component. This value of a describes numbers that have no roots of n, that is the rational numbers. All other values of a define numbers involving roots of n and are irrational.

Therefore the significance of a zero for $\frac{1}{n^s}\Big|^{s}_{(1-s)}$ is that we have identified the requirements (from the more general set of numbers n^s) where the imaginary part equals zero, *and* there are no irrational roots of n as well.

This leaves the integers n as the only permissible values remaining in the set since n^s is a single number, not a ratio. So as $\frac{1}{n^s}\Big|^{s}_{(1-s)} \to 0, \quad n^s \to n.$

This is the deep connection between the zeros of $\frac{1}{n^s}$ and the integers. The integers are part of a larger number set that includes the extra variables $n^{i\alpha}$ and n^a, but these zero out for the integers. These zeros indicate the exact values of a and α where the imaginary and irrational roots of the more general number set go to zero leaving only integers in the set.

Because the complex and irrational roots go to zero so we cannot detect these missing "ghost" dimensions. Therefore we need to map the larger domain over an interval around such a point, not at that point.

The components of the zeta function, that is the individual $\frac{1}{n^s}$, give definition as to what it means to be an integer. An integer is a single number (not a ratio) that has no imaginary component and is not a square root

(or other root) of n, and this corresponds with for $\frac{1}{n^s}\Big|_{(1-s)}^{s} = 0$. Additionally the sum of $\frac{1}{n^s}\Big|_{(1-s)}^{s}$ will contain the signatures of all the underlying n since each $\left[\mathrm{Im}\left(\frac{1}{n^{i\alpha}}\right)\right]$ value is irrational will always remain distinguishable from all the others.

This model also provides a more fundamental relationship between the prime and composite numbers. The reciprocal α_n values of the primes add to form those of the composites, and since these are irrational each composite carries the signature of all of its underlying primes, as per (6.13).

V. A Solution to the Riemann Hypothesis

The earlier mathematicians intuitively knew there was a connection between $\frac{1}{n^s}$ and the integers and this led to their fascination with these equations. It is in this light that we pose our "proof" of the Riemann Hypothesis. Our model addresses many of the fundamental questions and properties of integers, primes, and composites; it provides an association for a and α in that they determine the irrational roots and complex variables in a more general number system; and in the spirit of Riemann it provides a connection between the zeros of $\frac{1}{n^s}$ and the integers and shows that all zeros must occur at $a = 1/2$. The model uses a transformational mapping similar to Riemann's functional equation. Did we really prove the Hypothesis, and if so, in a way that satisfies the scientific community? We cannot tell, and therefore invite the community to comment.

A. Summation of n – A Proof of the Riemann Hypothesis

We now apply what we have learned from the sum $\frac{1}{2^s}+\frac{1}{3^s}\Big|^{s}_{(1-s)}$ to the general sum over n, $\sum_{n=1}^{\infty}\frac{1}{n^s}$, which is the zeta function (6.1). This follows the same line of reasoning as the previous section. We take (6.19) which was our starting point for $\frac{1}{2^s}+\frac{1}{3^s}\Big|^{s}_{(1-s)}$ and rewrite it in summation notation. We note that (6.19) is separated into a series of amplitude factors (real numbers) times a complex unit spinner.

$$\sum_{n=2}^{3}\frac{1}{n^s}\Big|^{s}_{(1-s)} = \frac{1}{2^s}+\frac{1}{3^s}\Big|^{s}_{(1-s)} = \left[\frac{1}{2^{(a+i\alpha)}}\right]-\left[\frac{1}{2^{(1-(a+i\alpha))}}\right] + \left[\frac{1}{3^{(a+i\alpha)}}\right]-\left[\frac{1}{3^{(1-(a+i\alpha))}}\right]$$

$$= \left[\left[\frac{1}{2^a}\cdot\frac{1}{2^{+i\alpha}}\right]-\left[\frac{1}{2^{(1-a)}}\cdot\frac{1}{2^{-i\alpha}}\right]\right] + \left[\left[\frac{1}{3^a}\cdot\frac{1}{3^{+i\alpha}}\right]-\left[\frac{1}{3^{(1-a)}}\cdot\frac{1}{3^{-i\alpha}}\right]\right] \qquad (6.22A)$$

Now expand the sum to the limit N and include $n = 1$:

$$\sum_{n=1}^{N}\frac{1}{n^s}\Big|^{s}_{(1-s)} = \frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\ldots\frac{1}{N^s}\Big|^{s}_{(1-s)} =$$

$$\left[\frac{1}{1^{(a+i\alpha)}}\right]-\left[\frac{1}{1^{(1-(a+i\alpha))}}\right] + \left[\frac{1}{2^{(a+i\alpha)}}\right]-\left[\frac{1}{2^{(1-(a+i\alpha))}}\right] + \left[\frac{1}{3^{(a+i\alpha)}}\right]-\left[\frac{1}{3^{(1-(a+i\alpha))}}\right] +$$

$$\left[\frac{1}{4^{(a+i\alpha)}}\right]-\left[\frac{1}{4^{(1-(a+i\alpha))}}\right] + \ldots\ldots \left[\frac{1}{N^{(a+i\alpha)}}\right]-\left[\frac{1}{N^{(1-(a+i\alpha))}}\right] \qquad (6.22B)$$

And separate this into a series of amplitude factors (real numbers) times a complex unit spinner.

$$\sum_{n=1}^{N}\frac{1}{n^s}\Big|^{s}_{(1-s)} = \left[\left[\frac{1}{1^a}\cdot\frac{1}{1^{+i\alpha}}\right]-\left[\frac{1}{1^{(1-a)}}\cdot\frac{1}{1^{-i\alpha}}\right]\right] + \left[\left[\frac{1}{2^a}\cdot\frac{1}{2^{+i\alpha}}\right]-\left[\frac{1}{2^{(1-a)}}\cdot\frac{1}{2^{-i\alpha}}\right]\right] +$$

$$\left[\left[\frac{1}{3^a}\cdot\frac{1}{3^{+i\alpha}}\right]-\left[\frac{1}{3^{(1-a)}}\cdot\frac{1}{3^{-i\alpha}}\right]\right] + \left[\left[\frac{1}{3^a}\cdot\frac{1}{3^{+i\alpha}}\right]-\left[\frac{1}{3^{(1-a)}}\cdot\frac{1}{3^{-i\alpha}}\right]\right] + \ldots\ldots \left[\left[\frac{1}{N^a}\cdot\frac{1}{N^{+i\alpha}}\right]-\left[\frac{1}{N^{(1-a)}}\cdot\frac{1}{N^{-i\alpha}}\right]\right]$$

(6.22C)

This is separated into two equations, one for the imaginary component and one for the real component:

$$[\text{Im}] \quad \left[\left[\frac{1}{1^a}\cdot \text{Im}\left(\frac{1}{1^{+i\alpha}}\right)\right]-\left[\frac{1}{1^{(1-a)}}\cdot \text{Im}\left(\frac{1}{1^{-i\alpha}}\right)\right]\right] + \left[\left[\frac{1}{2^a}\cdot \text{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right]-\left[\frac{1}{2^{(1-a)}}\cdot \text{Im}\left(\frac{1}{2^{-i\alpha}}\right)\right]\right] +$$
$$\left[\left[\frac{1}{3^a}\cdot \text{Im}\left(\frac{1}{3^{+i\alpha}}\right)\right]-\left[\frac{1}{3^{(1-a)}}\cdot \text{Im}\left(\frac{1}{3^{-i\alpha}}\right)\right]\right] + \left[\left[\frac{1}{4^a}\cdot \text{Im}\left(\frac{1}{4^{+i\alpha}}\right)\right]-\left[\frac{1}{4^{(1-a)}}\cdot \text{Im}\left(\frac{1}{4^{-i\alpha}}\right)\right]\right] +\ldots..$$
$$\left[\left[\frac{1}{N^a}\cdot \text{Im}\left(\frac{1}{N^{+i\alpha}}\right)\right]-\left[\frac{1}{N^{(1-a)}}\cdot \text{Im}\left(\frac{1}{N^{-i\alpha}}\right)\right]\right] \qquad (6.22D)$$

$$[\text{Re}] \quad \left[\left[\frac{1}{1^a}\cdot \text{Re}\left(\frac{1}{1^{+i\alpha}}\right)\right]-\left[\frac{1}{1^{(1-a)}}\cdot \text{Re}\left(\frac{1}{1^{-i\alpha}}\right)\right]\right] + \left[\left[\frac{1}{2^a}\cdot \text{Re}\left(\frac{1}{2^{+i\alpha}}\right)\right]-\left[\frac{1}{2^{(1-a)}}\cdot \text{Re}\left(\frac{1}{2^{-i\alpha}}\right)\right]\right] +$$
$$\left[\left[\frac{1}{3^a}\cdot \text{Re}\left(\frac{1}{3^{+i\alpha}}\right)\right]-\left[\frac{1}{3^{(1-a)}}\cdot \text{Re}\left(\frac{1}{3^{-i\alpha}}\right)\right]\right] + \left[\left[\frac{1}{4^a}\cdot \text{Re}\left(\frac{1}{4^{+i\alpha}}\right)\right]-\left[\frac{1}{4^{(1-a)}}\cdot \text{Re}\left(\frac{1}{4^{-i\alpha}}\right)\right]\right] +\ldots..$$
$$\left[\left[\frac{1}{N^a}\cdot \text{Re}\left(\frac{1}{N^{+i\alpha}}\right)\right]-\left[\frac{1}{N^{(1-a)}}\cdot \text{Re}\left(\frac{1}{N^{-i\alpha}}\right)\right]\right] \qquad (6.22E)$$

These can be written in summation notation:

$$[\text{Im}] \quad \sum_{n=1}^{N}\left[\left[\frac{1}{n^a}\cdot \text{Im}\left(\frac{1}{n^{+i\cdot\alpha}}\right)\right]-\left[\frac{1}{n^{(1-a)}}\cdot \text{Im}\left(\frac{1}{n^{-i\cdot\alpha}}\right)\right]\right] \qquad (6.23A)$$

$$[\text{Re}] \quad \sum_{n=1}^{N}\left[\left[\frac{1}{n^a}\cdot \text{Re}\left(\frac{1}{n^{+i\cdot\alpha}}\right)\right]-\left[\frac{1}{n^{(1-a)}}\cdot \text{Re}\left(\frac{1}{n^{-i\cdot\alpha}}\right)\right]\right] \qquad (6.23B)$$

With the substitutions $\left[\text{Im}\left(\frac{1}{2^{-i\alpha}}\right)\right] = -\left[\text{Im}\left(\frac{1}{2^{+i\alpha}}\right)\right]$ and $\left[\text{Re}\left(\frac{1}{2^{-i\alpha}}\right)\right] = -\left[\text{Re}\left(\frac{1}{2^{+i\alpha}}\right)\right]$ we obtain:

$$[\text{Im}] \quad \sum_{n=1}^{N}\left[\frac{1}{n^a}+\frac{1}{n^{1-a}}\right]\cdot\left[\text{Im}\left(\frac{1}{n^{-i\cdot\alpha}}\right)\right] \qquad (6.24A)$$

$$[\text{Re}] \quad \sum_{n=1}^{N}\left[\frac{1}{n^a}-\frac{1}{n^{1-a}}\right]\cdot\left[\text{Re}\left(\frac{1}{n^{-i\cdot\alpha}}\right)\right] \qquad (6.24B)$$

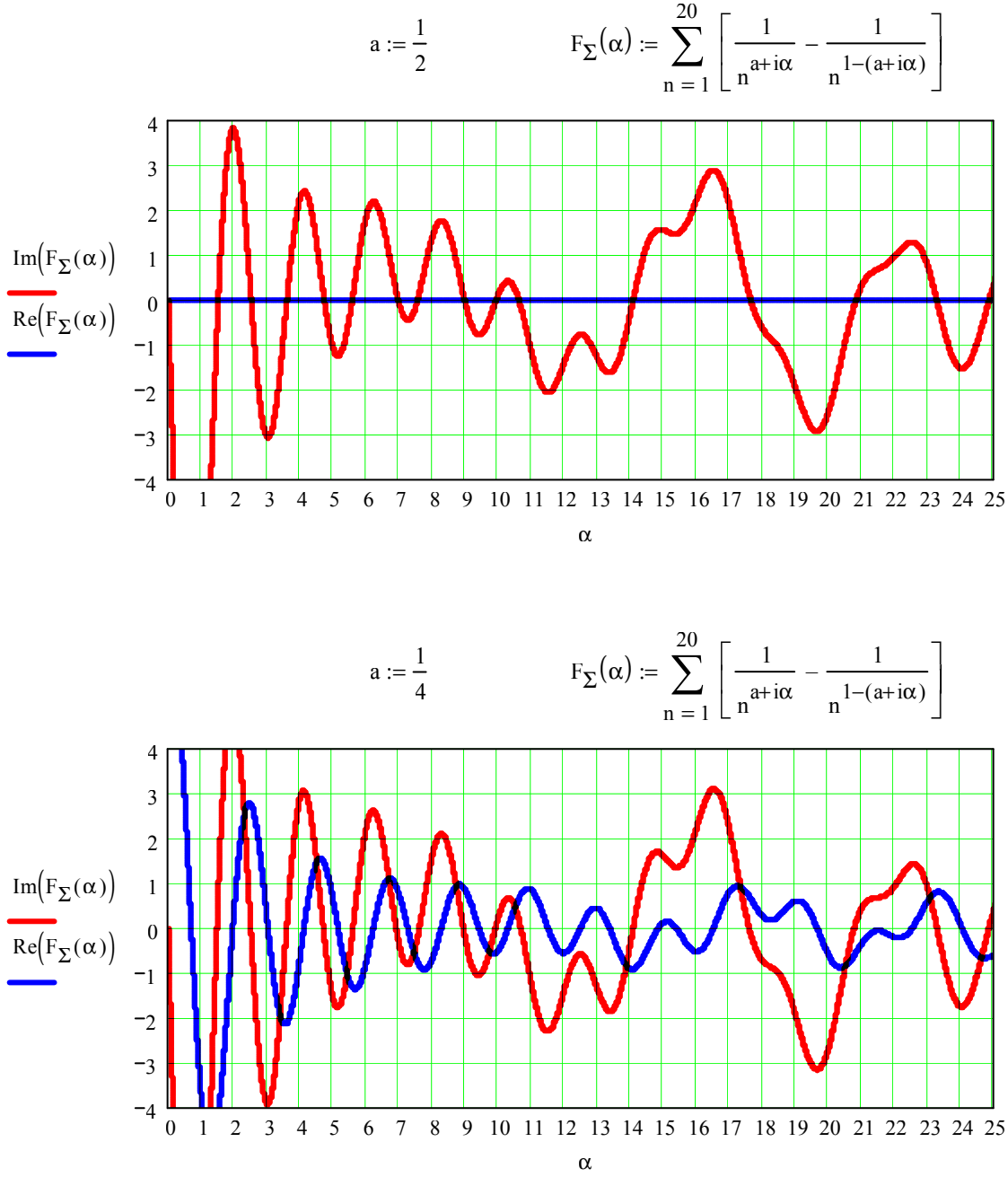


Figure 74 – The imaginary and real components of the sum of the first 20 terms. a = ½ (upper) & a = ¼ (lower).

In Figure 74 we plot the real and imaginary component sums first at $a = 1/2$ and then at $a = 1/4$ for the sum $n = 1...20$. When $a = 1/2$ the real equation sums to zero everywhere. This is because the real component subtracts to zero for *each* spinner pair as shown in Section IV.B. So the sum of these pairs will have no real component. The imaginary trace does have zeros at specific points as shown. There are specific values of α called α_Σ in which each spinner pair turns the exact amount so that the imaginary component of all of the spinner pairs sums to zero.

When $a = 1/4$ however, the real part is no longer zero everywhere. Now the real component has its own set of α values that result in the spinner pairs turning the exact amount so that real component of all the spinner pairs sums to zero.

The real and imaginary components of each spinner pair are out of phase. When one is zero the other is a maxima or minima. In Figure 74 Bottom we notice that the real component sum and imaginary component sum are generally out of phase as well. So intuitively it would be surprising to have both component sums zero at the same α value.

But in order for $\sum_{n=1}^{N} \frac{1}{n^s}\Big|_{(1-s)}^{s}$ to have a value of zero, both the real and the imaginary equations must equal zero. Again we ask the same question. If the imaginary equation equals zero, is a zero value for the real equation permissible (for the case $a \neq 1/2$) ? That is

[Im] For $a \neq 1/2$ If $$\sum_{n=1}^{N} \left[\frac{1}{n^a} + \frac{1}{n^{1-a}}\right] \cdot \left[\mathrm{Im}\left(\frac{1}{n^{+i\alpha}}\right)\right] = 0 \qquad (6.25A)$$

[Re] Can $$\sum_{n=1}^{N} \left[\frac{1}{n^a} - \frac{1}{n^{1-a}}\right] \cdot \left[\mathrm{Re}\left(\frac{1}{n^{+i\alpha}}\right)\right] = 0? \qquad (6.25B)$$

We know that $\left[\mathrm{Im}\left(\frac{1}{n^{+i\alpha}}\right)\right]$ and $\left[\mathrm{Re}\left(\frac{1}{n^{+i\alpha}}\right)\right]$ can be equated to the sides of a right triangle inscribed in the unit spinner and can be expressed as the sine and cosine of a common angle β_n. The sine and cosine of a common angle represent the square roots of two different numbers, so $\left[\mathrm{Im}\left(\frac{1}{n^{+i\alpha}}\right)\right]$ and $\left[\mathrm{Re}\left(\frac{1}{n^{+i\alpha}}\right)\right]$ are both irrational and of different bases.

If we let $A_n = \frac{1}{n^a}$ and $B_n = \frac{1}{n^{1-a}}$ we can rewrite (6.25 A&B) as

[Im] For $a \neq 1/2$ If $$\sum_{n=1}^{N} \left[A_n + B_n\right] \cdot \left[\sin(\beta_n)\right] = 0 \qquad (6.26A)$$

[Re] Can $$\sum_{n=1}^{N} \left[A_n - B_n\right] \cdot \left[\cos(\beta_n)\right] = 0? \qquad (6.26B)$$

If we expand this we get

$$\text{[Im]} \quad \text{If} \quad \left[A_1+B_1\right]\cdot\left[\sin(\beta_1)\right]+\left[A_2+B_2\right]\cdot\left[\sin(\beta_2)\right]+\left[A_3+B_3\right]\cdot\left[\sin(\beta_3)\right]+\left[A_4+B_4\right]\cdot\left[\sin(\beta_4)\right]+\ldots\left[A_N+B_N\right]\cdot\left[\sin(\beta_N)\right]=0 \tag{6.27A}$$

$$\text{[Re]} \quad \text{Can} \quad \left[A_1-B_1\right]\cdot\left[\cos(\beta_1)\right]+\left[A_2-B_2\right]\cdot\left[\cos(\beta_2)\right]+\left[A_3-B_3\right]\cdot\left[\cos(\beta_3)\right]+\left[A_4-B_4\right]\cdot\left[\cos(\beta_4)\right]+\ldots\left[A_N-B_N\right]\cdot\left[\cos(\beta_N)\right]=0? \tag{6.27B}$$

If we look at the imaginary equation we see that the amplitude factors $\left[A_n+B_n\right]$ are real but irrational and of different bases. These irrational numbers do not combine under addition, they remain separate. For this equation to zero each individual $\left[\sin(\beta_n)\right]$ must match with its corresponding $\left[A_n+B_n\right]$ negating its unique irrational base so that it can be summed in general, and summed to zero specifically. The set of β_n values is highly specific and is much like a key. It contains the match to every $\left[A_n+B_n\right]$ for this zero. All of the β_n are related by the single variable α. As α turns, all of the β_n all turn, but at different speeds:

$$\left[\text{Im}\left(\frac{1}{n^{+i\cdot\alpha}}\right)\right]=\left[\sin(\beta_n)\right] \tag{6.28A}$$

which can be rewritten to provide β_n in terms of α:

$$\beta_n = a\sin\left[\text{Im}\left(\frac{1}{n^{+i\alpha}}\right)\right] \tag{6.28B}$$

The value $\alpha=\alpha_\Sigma$ encodes the entire set of $\left[\sin(\beta_n)\right]$ that match the set of $\left[A_n+B_n\right]$ exactly so that the N products of $\left[A_n+B_n\right]\cdot\left[\sin(\beta_n)\right]$ sum to zero. This α value is also highly specific. It encodes the entire set of β_n values for this zero.

Can we expect the cosines of the same β_n angles to zero with $\left[A_n-B_n\right]$ in the same way ?

No. The sine of each β_n angle is irrational and matches multiplying with an irrational amplitude factor $\left[A_n+B_n\right]$ in (6.27A). When the cosines of these same β_n angles (6.27B) try to match up multiplying with a linear combination of the same irrational amplitudes A_n and B_n and sum to zero, this is not permissible. That is because the cosine is also irrational and is of a different base than the sine and involves the square root of a different number than the sine. If the set $\left[A_n+B_n\right]$ matches with the set $\left[\sin(\beta_n)\right]$ and sums to zero, then it cannot zero with $\left[\cos(\beta_n)\right]$, nor can any (rational) linear combinations of A_n and B_n. Therefore:

[Im] If $\sum_{n=1}^{N} \left[A_n + B_n\right] \cdot \left[\sin(\beta_n)\right] = 0$ (6.29A)

[Re] Then $\sum_{n=1}^{N} \left[a \cdot A_n + b \cdot B_n\right] \cdot \left[\cos(\beta_n)\right] \neq 0$ (6.29B)

where a and b are non-zero integers or integer ratios (rational numbers). So no linear combinations are permissible. For $\left[A_n - B_n\right]$, $b_n = -1$. So this *is* a linear combination and therefore a zero value for both equations is forbidden. Therefore:

[Im] If $\sum_{n=1}^{N} \left[A_n + B_n\right] \cdot \left[\sin(\beta_n)\right] = 0$ (6.30A)

[Re] Then $\sum_{n=1}^{N} \left[A_n - B_n\right] \cdot \left[\cos(\beta_n)\right] \neq 0$ (6.30B)

Substituting back to our original form

[Im] If $\sum_{n=1}^{N} \left[\frac{1}{n^a} + \frac{1}{n^{1-a}}\right] \cdot \left[\mathrm{Im}\left(\frac{1}{n^{+i\alpha}}\right)\right] = 0$ (6.31A)

[Re] Then $\sum_{n=1}^{N} \left[\frac{1}{n^a} - \frac{1}{n^{1-a}}\right] \cdot \left[\mathrm{Re}\left(\frac{1}{n^{+i\alpha}}\right)\right] \neq 0$ (6.31B)

Therefore when $a \neq 1/2$, the imaginary equation will sum to zero at certain α values, but it is not permissible for the real equation also sum to zero at the same α values. So the real and imaginary equations cannot zero at the same time. These were the separated into components from $\sum_{n=1}^{N} \frac{1}{n^s}\Big|_{(1-s)}^{s} = 0$. The recombination cannot equal zero if both parts are not zero. Therefore

When $a \neq 1/2$, $\sum_{n=1}^{N} \frac{1}{n^s}\Big|_{(1-s)}^{s} \neq 0$ (6.32)

And there are no zeros when $a \neq 1/2$.

When $a = 1/2$ the real component cancels out for each spinner pair. We associate this action with the amplitude factor subtracting to zero. For (6.31B) the amplitude factor is:

$$\left[\frac{1}{n^a} - \frac{1}{n^{1-a}}\right] = \left[\frac{1}{n^{1/2}} - \frac{1}{n^{1-1/2}}\right] = \left[\frac{1}{n^{1/2}} - \frac{1}{n^{1/2}}\right] = 0 \quad (6.33)$$

and therefore the real equation equals zero everywhere. The imaginary equation has solutions at $\alpha = \alpha_\Sigma$ where the sum of the *N* imaginary components sums to zero as already discussed. So zero values are permissible for $a = 1/2$.

Therefore all zeros of $\sum_{n=1}^{N} \frac{1}{n^s}\Big|_{(1-s)}^{s}$ occur at $a = 1/2$, as they are forbidden when $a \neq 1/2$ and permissible at $a = 1/2$.

B. Extension of Sum to Infinity

The extension of $\sum_{n=1}^{N} \frac{1}{n^s}\Big|_{(1-s)}^{s}$ to $\sum_{n=1}^{\infty} \frac{1}{n^s}\Big|_{(1-s)}^{s}$ is straightforward. With the summation to *N*, it does not matter how large *N* is or how small $\frac{1}{n}$ becomes, each $\frac{1}{n}$ will be distinguishable from all the others in the sum because they are all irrational and of different bases and therefore they cannot combine through addition. So the unique irrational signatures of every $\frac{1}{n}$ is present in the sum. This is true no matter how small $\frac{1}{n}$ is, even if it is infinitesimally small. The mechanics of summing to *N* and to infinity are the same.

Therefore we again look at the two cases $a = 1/2$ and $a \neq 1/2$, now for the zeros of $\sum_{n=1}^{\infty} \frac{1}{n^s}\Big|_{(1-s)}^{s}$.

$\sum_{n=1}^{\infty} \frac{1}{n^s}\Big|_{(1-s)}^{s}$ can be separated into the imaginary and real equations:

[Im] $$\sum_{n=1}^{\infty} \left[\frac{1}{n^a} + \frac{1}{n^{1-a}}\right] \cdot \left[\mathrm{Im}\left(\frac{1}{n^{+i\cdot\alpha}}\right)\right] \qquad (6.34A)$$

[Re] $$\sum_{n=1}^{\infty} \left[\frac{1}{n^a} - \frac{1}{n^{1-a}}\right] \cdot \left[\mathrm{Re}\left(\frac{1}{n^{+i\cdot\alpha}}\right)\right] \qquad (6.34B)$$

When $a = 1/2$, the amplitude factor $\left[\frac{1}{n^a} - \frac{1}{n^{1-a}}\right] = 0$ and so the real equation equals zero everywhere. Equivalently we can say that at $a = 1/2$ the real part subtracts out of each spinner pair so there are no real numbers present when summing the pairs. The imaginary equation has solutions at $\alpha = \alpha_\Sigma$ which is the α value where the sum of the imaginary components equals zero. We would expect the change in α_Σ to be small for large

n, yet each *n* will always leave its trace through its unique irrational signature. So zero values are permissible for $a = 1/2$.

For $a \neq 1/2$ we again ask if it is permissible for the real equation to zero when the imaginary does:

[Im] For $a \neq 1/2$ If $$\sum_{n=1}^{\infty} \left[\frac{1}{n^a} + \frac{1}{n^{1-a}}\right] \cdot \left[\mathrm{Im}\left(\frac{1}{n^{+i\alpha}}\right)\right] = 0 \qquad (6.35A)$$

[Re] Can $$\sum_{n=1}^{\infty} \left[\frac{1}{n^a} - \frac{1}{n^{1-a}}\right] \cdot \left[\mathrm{Re}\left(\frac{1}{n^{+i\alpha}}\right)\right] = 0? \qquad (6.35B)$$

Again we see that $\left[\mathrm{Im}\left(\frac{1}{n^{+i\alpha}}\right)\right]$ and $\left[\mathrm{Re}\left(\frac{1}{n^{+i\alpha}}\right)\right]$ are the square roots of different numbers so they are irrational and of different bases. Therefore if the imaginary equation equals zero, then no (rational) linear combination of the amplitude factor $\frac{1}{n^a}\Big|^{a}_{(1-a)} = \left[\frac{1}{n^a} + \frac{1}{n^{1-a}}\right]$ can result in a zero value for the real equation.

Then with $A_n = \frac{1}{n^a}$ and $B_n = \frac{1}{n^{1-a}}$, and a and b non-zero integers or integer ratios (rational numbers) we can rewrite this as:

[Im] If $$\sum_{n=1}^{\infty} \left[A_n + B_n\right] \cdot \left[\mathrm{Im}\left(\frac{1}{n^{+i\alpha}}\right)\right] = 0 \qquad (6.36A)$$

[Re] Then $$\sum_{n=1}^{\infty} \left[a \cdot A_n + b \cdot B_n\right] \cdot \left[\mathrm{Re}\left(\frac{1}{n^{+i\alpha}}\right)\right] \neq 0 \qquad (6.36B)$$

But $\left[\frac{1}{n^a} - \frac{1}{n^{1-a}}\right]$ *is* a linear combination of $\left[\frac{1}{n^a} + \frac{1}{n^{1-a}}\right]$, so it is not permissible for the real equation to zero when the imaginary one does. Therefore:

[Im] If $$\sum_{n=1}^{\infty} \left[\frac{1}{n^a} + \frac{1}{n^{1-a}}\right] \cdot \left[\mathrm{Im}\left(\frac{1}{n^{+i\alpha}}\right)\right] = 0 \qquad (6.37A)$$

[Re] Then $$\sum_{n=1}^{\infty} \left[\frac{1}{n^a} - \frac{1}{n^{1-a}}\right] \cdot \left[\mathrm{Re}\left(\frac{1}{n^{+i\alpha}}\right)\right] \neq 0 \qquad (6.37B)$$

And in the case of $a = 1/2$ the real equation is zero everywhere so only the imaginary equation is of interest.

At $a = 1/2$, (6.37A) simplifies to:

[Im] At $a = 1/2$ $$\sum_{n=1}^{\infty} \left[\frac{2}{n^a}\right] \cdot \left[\mathrm{Im}\left(\frac{1}{n^{+i\alpha}}\right)\right] = 0 \qquad (6.38A)$$

[Re] $$= 0 \quad \text{Everywhere} \qquad (6.38B)$$

which has zeros at α_Σ. This is the only way that both the imaginary and real equations can both zero simultaneously which is required for a zero. So $\sum_{n=1}^{\infty} \frac{1}{n^s}\Big|_{(1-s)}^{s} = 0$ at $a = 1/2$ and $\alpha = \alpha_\Sigma$ only.

Therefore all zeros of $\sum_{n=1}^{\infty} \frac{1}{n^s}\Big|_{(1-s)}^{s}$ occur at $a = 1/2$, as they are forbidden when $a \neq 1/2$ and permissible at $a = 1/2$.

How is this result for $\sum_{n=1}^{\infty} \frac{1}{n^s}\Big|_{(1-s)}^{s}$ applicable to the zeta function $\sum_{n=1}^{\infty} \frac{1}{n^s}$?

$\sum_{n=1}^{\infty} \frac{1}{n^s}$ cannot zero at a single point because the imaginary and real components cannot have zero value simultaneously. For each *n*, the imaginary and real component are out of phase by one quarter cycle so these cannot be equal to zero at the same α value. Nor is it possible for a component from one *n* to add with that of another *n* and sum to zero. The component amplitudes for each *n* are irrational and of different bases, so these cannot sum to zero.

The transformational mapping $\sum_{n=1}^{\infty} \frac{1}{n^s}\Big|_{(1-s)}^{s}$ describes the result of an action $A \to B$ where $A = 1 - s$ to $B = s$:

$$\sum_{n=1}^{\infty} \frac{1}{n^s}\Big|_{(1-s)}^{s} = \int_{1-s}^{s} \left[\frac{d}{ds} \cdot \frac{1}{n^s}\right] \cdot ds = \int_{1-s}^{s} \left[\frac{1}{n^s} \cdot \ln(n)\right] \cdot ds \qquad (6.39)$$

Figure 74½ is a graphical model of this action. In the complex plane the action is $(-\alpha) \rightarrow (+\alpha)$, and in the *a* dimension it is $(1-s) \rightarrow (s)$. The *a* dimension is depicted as the magnitude of the complex spinner. Thus the action shown is the radius of the spinner transitioning $\frac{1}{n^{1-s}} \rightarrow \frac{1}{n^{s}}$. The action occurs around the values $\alpha = 0$ and $a = \frac{1}{2}$ which is where the integers occur. Therefore the transformational mapping $\sum_{n=1}^{\infty} \frac{1}{n^s}\Big|_{(1-s)}^{s}$ describes the action of $\sum_{n=1}^{\infty} \frac{1}{n^s}$ over an interval around this point.

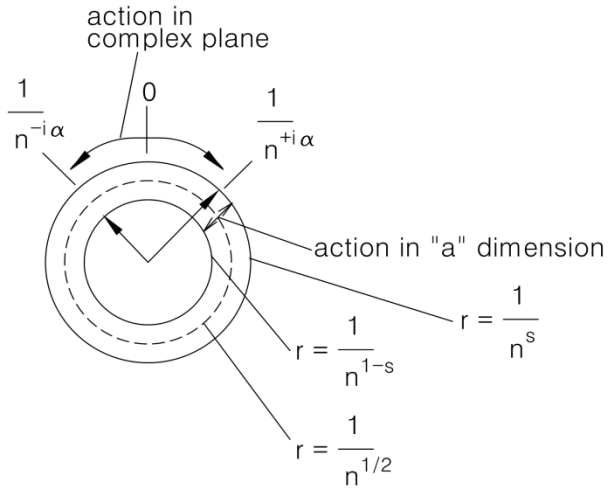


Figure 74½ - $\frac{1}{n^s}\Big|_{(1-s)}^{s}$ maps the action $(-\alpha) \rightarrow (+\alpha)$ in the complex plane and $\frac{1}{n^{1-s}} \rightarrow \frac{1}{n^{s}}$ in the *a* dimension.

This completes the discussion on the mechanics, but the proof is incomplete without showing equivalence of the transformational mapping through modal harmonics. We must show that $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is equivalently mapped as $\sum_{n=1}^{\infty} \frac{1}{n^s}\Big|_{(1-s)}^{s}$, and that $a = \frac{1}{2}$ for zeros also applies in real space.

VI. Modal Harmonic Analysis

Thus far we have built a model based on the individual $\frac{1}{n}$ raised to the complex power *s* from the wave function each of these $\frac{1}{n^s}\Big|_{1-s}^{s}$ creates. Using a model with two opposing spinners it was shown that $\sum_{n=1}^{\infty}\frac{1}{n^s}\Big|_{1-s}^{s}$ can be separated into real and imaginary component equations, with the real component equal to zero at $a = \frac{1}{2}$ and the imaginary component equal to zero at $\alpha = \alpha_{\Sigma}$. These equations provide a description of the mechanics based on the model with opposing spinner pairs.

For the Modal Harmonic Analysis, we want to map $\sum_{n=1}^{\infty}\frac{1}{n^s}\Big|_{1-s}^{s}$ in harmonic space to demonstrate graphical equivalence, and to verify that zeros are forbidden when $a \neq \frac{1}{2}$.

To proceed in this evaluation we must first develop the Range Transform and the Reciprocal Range Transform.

A. The Unity, Range, & Reciprocal Range Transforms

Previously we indicated that a function could be broken down into composite waves ordered harmonically each with an amplitude factor that operates as a coding function. We recall that in our work with π we were able to create a linear function over an interval created from a series of wave functions, and with normalization constants a function was created that was graphically equivalent to that of measuring the circumference of a circle unrolled to form a linear map. The coefficients of this series are in fact the Gregory Series for $\pi/4$. An equivalence relation for π was determined by summing the coefficients of the series that solved the differential equation created by the General Transformation Equation (GTE). Based on this knowledge we can now: 1) Transform the linear function to map *n* and its reciprocal into a series of wave functions with coefficients, 2) Use this to map the zeta function in harmonic space 3) Confirm that zeros are forbidden when $a \neq ½$ and 4) Show that the coefficients which solve the GTE sum to form the zeta function.

Using the GTE:
$$\int_{A}^{B} f(x)dx = \int_{A}^{B} F(X)dX \tag{6.41}$$

we require equivalence under transformation between real space or x space (left side), and harmonic space or X space (right side). Thus a linear graph of the points in x space can be remapped into the harmonic X space, where the X space map is composed of a series of wave functions. In this case $f(x) = 1/2$. See Figure 75A.

For the left side of the GTE:
$$\int_{A}^{B} f(x)dx = \int_{-n}^{+n} \frac{1}{2}\cdot dx = \frac{1}{2}x\Big|_{-n}^{+n} = n \tag{6.42}$$

Based on the π equations we will try $F(X)=\sum_{m=1}^{\infty}(e^{i\pi mX})$. For the right side of the GTE our trial transform is:

$$T(X) = \int_{-X}^{+X}\sum_{m=1}^{\infty}(e^{i\pi mX})\cdot dX = \sum_{m=1}^{\infty}\frac{e^{i\pi mX}-e^{-i\pi mX}}{im\pi} \tag{6.43}$$

From Figure 75A we see that Equation (6.43) needs to be normalized. We will try $F(X)=-\sum_{m=1}^{\infty}(e^{i\pi m(X-1)})$.

And $$U(X) = -\int_{-X}^{+X}\sum_{m=1}^{\infty}(e^{i\pi m(X-1)})\cdot dX = -\sum_{m=1}^{\infty}\frac{e^{i\pi m(X-1)}-e^{i\pi m(-X-1)}}{im\pi} \tag{6.44}$$

Equation (6.44) is graphically equivalent to the linear function, but is limited in range from -1 to +1 and is cyclical. (Fig. 75B). We call this the ***Linear Unit Transformation,*** $U(X)$.

$$T(X) := \sum_{m=1}^{100}\frac{e^{i\cdot\pi\cdot m X}-e^{-i\cdot\pi\cdot m\cdot X}}{im\cdot\pi} \qquad U(X) := -\sum_{m=1}^{100}\frac{e^{i\cdot\pi\cdot m(X-1)}-e^{i\cdot\pi\cdot m\cdot(-X-1)}}{im\cdot\pi}$$

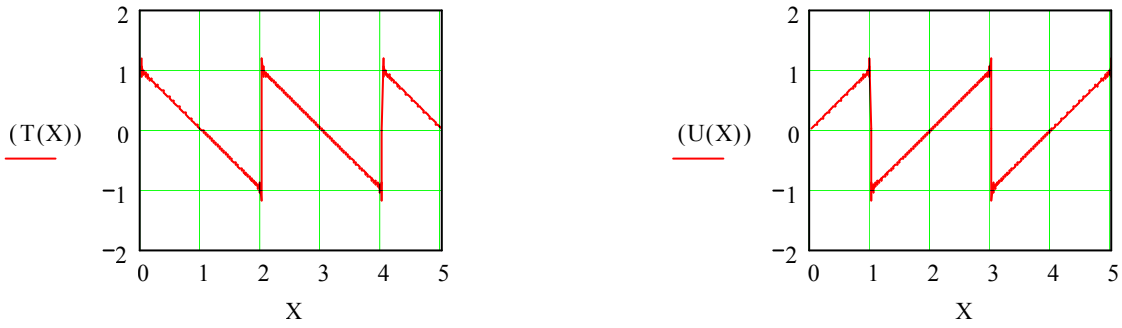


Figure 75A&B – It was found in the work on π that a linear function in real space could be transformed into a series of wave functions in harmonic space, and that this could be normalized to create a linear function over an interval. In 75A the trial guess of T(X) yields a linear function requiring normalization. 75B is normalized, but it is limited to the interval -1 to +1 and is cyclical. This is called the Linear Unit Transformation.

To represent the integers n in the zeta function we cannot be constrained to the interval -1 to +1. We wish to expand the Linear Unit Transformation to a range limit R. This expansion of the linear unit function of (6.44) to the range limit R where $R > X$ results in the ***Range Transformation,*** $RT(X)$. (Figure 76):

$$RT(X) = -\int_{-X}^{+X}\sum_{m=1}^{\infty}(e^{i\pi m(X-R)/R})\cdot dX = -R\cdot\sum_{m=1}^{\infty}\frac{e^{i\pi m(X-R)/R}-e^{i\pi m(-X-R)/R}}{im\pi} \tag{6.45}$$

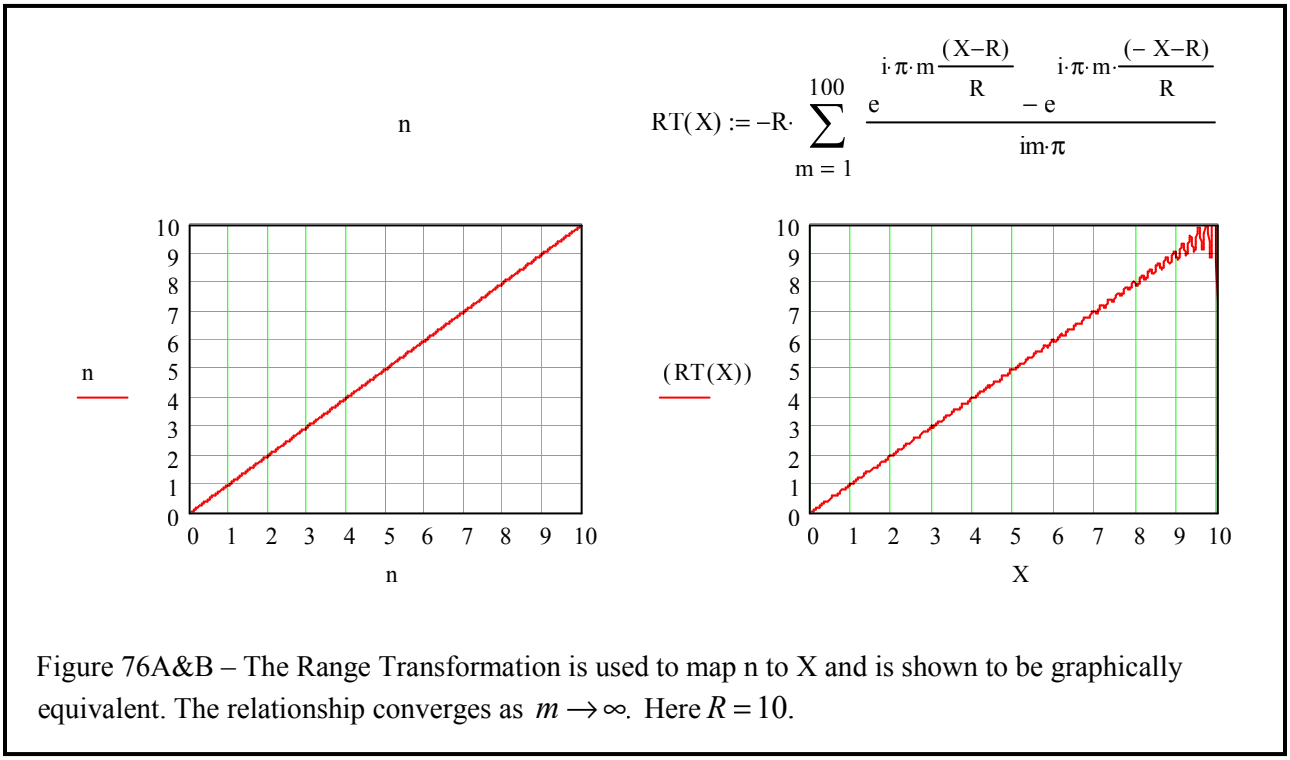


Figure 76A&B – The Range Transformation is used to map n to X and is shown to be graphically equivalent. The relationship converges as $m \to \infty$. Here $R = 10$.

The Range Transform an equivalent mapping in harmonic space of the function $f(n) = n$ in real space over the range. In Figure 76 these are shown to be graphically equivalent. However we need to map $f(n) = \frac{1}{n}$. This requires the ***Reciprocal Range Transform*** which is the reciprocal of the Range Transform. (See Figure 77):

$$RRT(X) = \left[-\int_{-X}^{+X} \sum_{m=1}^{\infty} \left(e^{i\pi m(X-R)/R} \right) \cdot dX \right]^{-1} = \left[-R \cdot \sum_{m=1}^{\infty} \frac{e^{i\pi m(X-R)/R} - e^{i\pi m(-X-R)/R}}{im\pi} \right]^{-1} \quad (6.46)$$

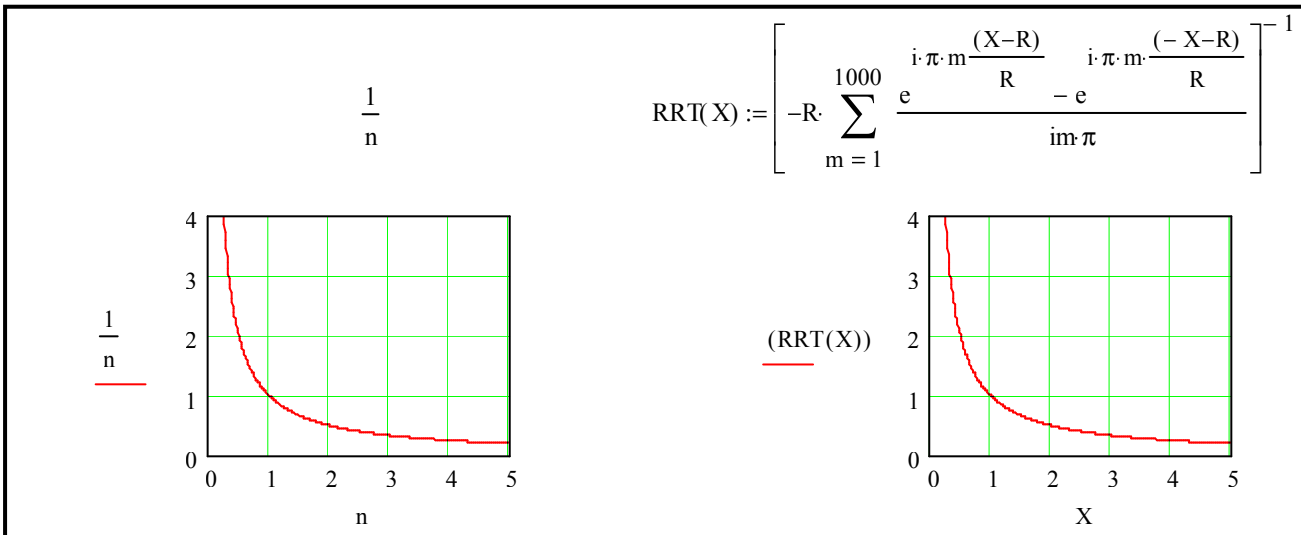


Figure 77 – The Reciprocal Range Transform is graphically equivalent to 1/n.

$m := 1, 2 .. 25 \qquad X := 5 \qquad R := 12$

m =	***Spinner 1*** $\frac{e^{\frac{i\cdot\pi\cdot m\cdot(X-R)}{R}}}{i} =$	***Spinner 2*** $\frac{e^{\frac{i\cdot\pi\cdot m\cdot(-X-R)}{R}}}{i} =$	***Difference/2*** $\frac{e^{\frac{i\cdot\pi\cdot m\cdot(X-R)}{R}} - e^{\frac{i\cdot\pi\cdot m\cdot(-X-R)}{R}}}{2\cdot i}$
1	-0.966+0.259i	0.966+0.259i	-0.966
2	0.5+0.866i	-0.5+0.866i	0.5
3	0.707-0.707i	-0.707-0.707i	0.707
4	-0.866-0.5i	0.866-0.5i	-0.866
5	-0.259+0.966i	0.259+0.966i	-0.259
6	1	-1	1
7	-0.259-0.966i	0.259-0.966i	-0.259
8	-0.866+0.5i	0.866+0.5i	-0.866
9	0.707+0.707i	-0.707+0.707i	0.707
10	0.5-0.866i	-0.5-0.866i	0.5
11	-0.966-0.259i	0.966-0.259i	-0.966
12	i	i	0
13	0.966-0.259i	-0.966-0.259i	0.966
14	-0.5-0.866i	0.5-0.866i	-0.5
15	-0.707+0.707i	0.707+0.707i	-0.707
16	0.866+0.5i	-0.866+0.5i	0.866

m =	$\frac{e^{\frac{i\cdot\pi\cdot m\cdot(X-R)}{R}}}{i} =$	$\frac{e^{\frac{i\cdot\pi\cdot m\cdot(-X-R)}{R}}}{i} =$	$\frac{e^{\frac{i\cdot\pi\cdot m\cdot(X-R)}{R}} - e^{\frac{i\cdot\pi\cdot m\cdot(-X-R)}{R}}}{2\cdot i}$
17	0.259-0.966i	-0.259-0.966i	0.259
18	-1	1	-1
19	0.259+0.966i	-0.259+0.966i	0.259
20	0.866-0.5i	-0.866-0.5i	0.866
21	-0.707-0.707i	0.707-0.707i	-0.707
22	-0.5+0.866i	0.5+0.866i	-0.5
23	0.966+0.259i	-0.966+0.259i	0.966
24	-i	-i	0
25	-0.966+0.259i	0.966+0.259i	-0.966
26	0.5+0.866i	-0.5+0.866i	0.5
27	0.707-0.707i	-0.707-0.707i	0.707

$$\left[-R\cdot \sum_{m=0}^{10000} \frac{e^{i\cdot\pi\cdot m\cdot\frac{(X-R)}{R}} - e^{i\cdot\pi\cdot m\cdot\frac{(-X-R)}{R}}}{i\cdot m\cdot\pi} \right]^{-1} = 0.200$$

Figure 78 – The Reciprocal Range Formula can be modeled as the difference of two opposing spinners. The real parts of each cancel out and the imaginary parts double. The difference is divided by 2 for normalization to the unit circle which gives terms we are familiar with. The calculation shows the reciprocal of the sum is 1/*n*.

In Figure 78 we examine the series for $n = 5$ and $R = 12$. It is apparent that that the unit circle breaks into 24 segments. As expected, the real component of each spinner cancels out, and the imaginary components double. We divide our difference by 2 as a normalization factor to give this value a location on the unit circle. There are familiar terms at multiples of $\pi/6$ (m=2,4,8, 10…) and $\pi/4$ (m=3,9, 15…) for example.

These terms do in fact sum to 1/*n*. Other choices of *n* and *R* will create a different numerical series which still sums to 1/*n*. Our selection of $n = 5$ and $R = 12$ is meant to illustrate a random case. As will be seen, simpler but equivalent series can be obtained through the selection of variables.

The Reciprocal Range Transform is the transformation of $\frac{1}{n}$ and the two are graphically equivalent (Figure 77). It maps the function $\frac{1}{n}$ in real space into a series of component waves in harmonic space. The zeta function can be formed through algebraic operation and summation of $\frac{1}{n}$. So we can substitute our transformation of $\frac{1}{n}$ into the zeta function to obtain an equivalence for the zeta function in harmonic space.

The Reciprocal Range Transform is an equivalence relation for $\frac{1}{n}$:

$$\frac{1}{n} = \left[-\int_{-X}^{+X} \sum_{m=1}^{\infty} (e^{i\pi m(\mathrm{X}-R)/R}) \cdot dX \right]^{-1} = \left[-R \cdot \sum_{m=1}^{\infty} \frac{e^{i\pi m(\mathrm{X}-R)/R} - e^{i\pi m(-X-R)/R}}{im\pi} \right]^{-1} \tag{6.47}$$

Next we substitute *N* for *X*. This makes the equation easier to understand since *N* will be the number in harmonic space that corresponds with *n* in real space.

$$\frac{1}{n} = \left[-\int_{-N}^{+N} \sum_{m=1}^{\infty} (e^{i\pi m(N-R)/R}) \cdot dN \right]^{-1} = \left[-R \cdot \sum_{m=1}^{\infty} \frac{e^{i\pi m(N-R)/R} - e^{i\pi m(-N-R)/R}}{im\pi} \right]^{-1} \tag{6.48}$$

If we raise both sides of (7.37) to the *s* power we obtain:

$$\frac{1}{n^s} = \frac{1}{\left[-\int_{-N}^{+N} \sum_{m=1}^{\infty} (e^{i\pi m(N-R)/R}) \cdot dN \right]^s} = \frac{1}{\left[-R \cdot \sum_{m=1}^{\infty} \frac{e^{i\pi m(N-R)/R} - e^{i\pi m(-N-R)/R}}{im\pi} \right]^s} \tag{6.49}$$

And the zeta function is the sum of these over n:

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s} \;=\; \sum_{N=1}^{\infty}\frac{1}{\left[-\int_{-N}^{+N}\sum_{m=1}^{\infty}(e^{i\pi m(N-R)/R})\cdot dN\right]^s} \;=\; \sum_{N=1}^{\infty}\frac{1}{\left[-R\cdot\sum_{m=1}^{\infty}\frac{e^{i\pi m(N-R)/R}-e^{i\pi m(-N-R)/R}}{im\pi}\right]^s} \tag{6.50}$$

We want to evaluate the zeta function over the interval $1-s$ to s. Therefore:

$$\zeta(s)\Big|_{1-s}^{s}=\sum_{n=1}^{\infty}\frac{1}{n^s}\Big|_{1-s}^{s} \;=$$

$$\sum_{N=1}^{\infty}\frac{1}{\left[-\int_{-N}^{+N}\sum_{m=1}^{\infty}(e^{i\pi m(N-R)/R})\cdot dN\right]^s}\Bigg|_{1-s}^{s} \;=\; \sum_{N=1}^{\infty}\frac{1}{\left[-R\cdot\sum_{m=1}^{\infty}\frac{e^{i\pi m(N-R)/R}-e^{i\pi m(-N-R)/R}}{im\pi}\right]^s}\Bigg|_{1-s}^{s} \tag{6.51}$$

And performing this evaluation and subsequent expansion

$$\zeta(s)\Big|_{1-s}^{s}=\sum_{n=1}^{\infty}\frac{1}{n^s}\Big|_{1-s}^{s} \;= \tag{6.52A}$$

$$\sum_{N=1}^{\infty}\frac{1}{\left[-\int_{-N}^{+N}\sum_{m=1}^{\infty}(e^{i\pi m(N-R)/R})\cdot dN\right]^s}-\frac{1}{\left[-\int_{-N}^{+N}\sum_{m=1}^{\infty}(e^{i\pi m(N-R)/R})\cdot dN\right]^{1-s}} \;= \tag{6.52B}$$

$$\sum_{N=1}^{\infty}\frac{1}{\left[-R\cdot\sum_{m=1}^{\infty}\frac{e^{i\pi m(N-R)/R}-e^{i\pi m(-N-R)/R}}{im\pi}\right]^s}-\frac{1}{\left[-R\cdot\sum_{m=1}^{\infty}\frac{e^{i\pi m(N-R)/R}-e^{i\pi m(-N-R)/R}}{im\pi}\right]^{1-s}} \tag{6.52C}$$

In Figure 79A we show that $\frac{1}{n^s}\Big|_{1-s}^{s}$ is graphically equivalent to the transform of (7.42C). It can be seen that the transform follows the same rule forbidding zeros when $a \neq 1/2$. The real and imaginary part of the transform cannot zero at the same time when $a \neq 1/2$ because they are both irrational and of different bases. When $a = 1/2$ zeros are permissible. The real component is zero everywhere and the imaginary part does have zeros (Fig 79B).

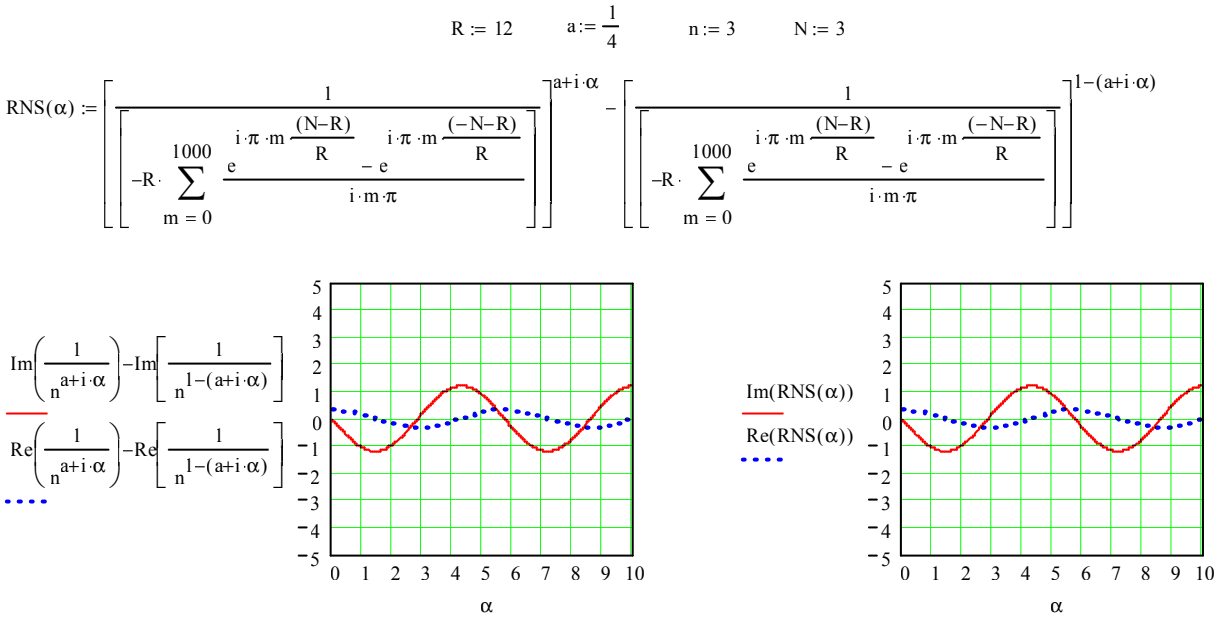


Figure 79A – $\left.\frac{1}{n^s}\right|_{1-s}^{s}$ is graphically equivalent to the expansion of (6.52C). The graph is for $a = 1/4$ and $\alpha = 3$. Zeros are forbidden because the real and imaginary parts are irrational and of different bases, so their zeros cannot occur at the same values, for any *n* or sum of *n*.

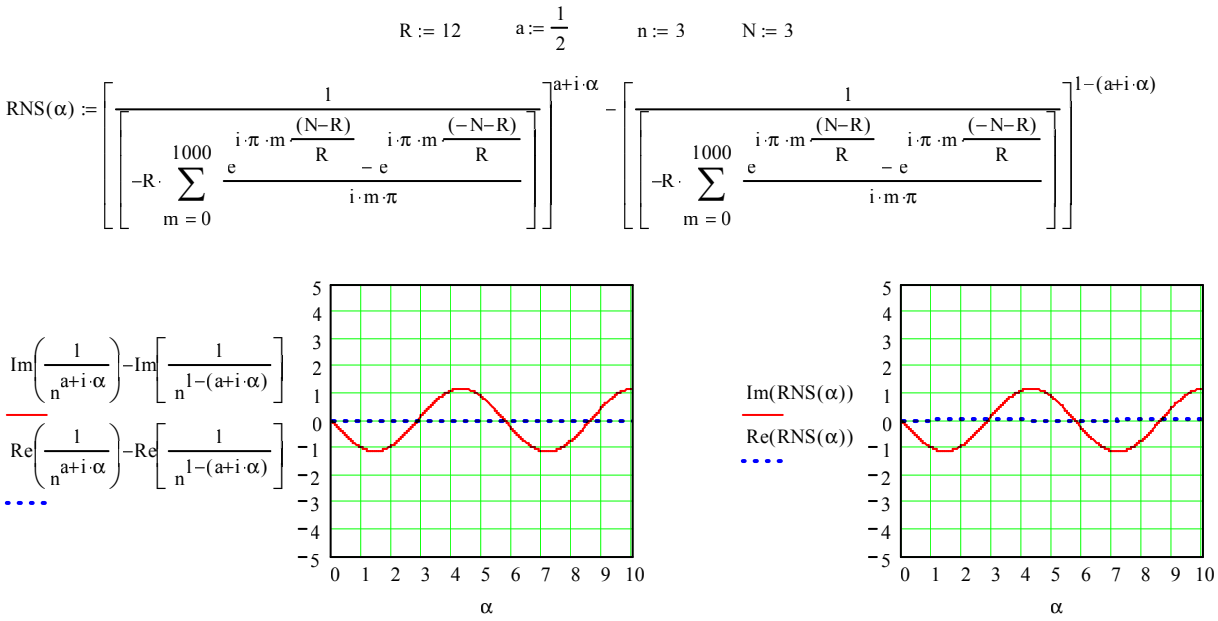


Figure 79B – At $a = 1/2$ zeros are permissible. The real component is zero everywhere and the imaginary part does have zeros.

(6.52C) contains the term $e^{i\pi m(N-R)/R} - e^{i\pi m(-N-R)/R}$ embedded in it. ***We recognize this as the difference of two opposing spinners***, ***and that it can be expressed as a simple series over m similar to the Gregory Series.***

When the series $\left[-R \cdot \sum_{m=1}^{\infty} \frac{e^{i\pi m(N-R)/R} - e^{i\pi m(-N-R)/R}}{im\pi} \right]^{-1}$ is summed, the result will be $\frac{1}{N}$ for all R so long as $R > N$ since this is the Reciprocal Range Formula. Therefore after integration we can select any value for R after integration so long as $R > N$. We will choose $R = 2N$ for simplicity in the tables. For the exponents in (7.42C):

With $R = 2N$ $\quad (N-R)/R = (N-2N)/2N = -\frac{1}{2}.$

And $(-N-R)/R = (-N-2N)/2N = -\frac{3}{2}.$

And (7.42) simplifies to:

$$\zeta(s)\Big|_{1-s}^{s} = \sum_{N=1}^{\infty} \frac{1}{\left[-2N \cdot \sum_{m=1}^{\infty} \frac{e^{i\pi m(-1/2)} - e^{i\pi m(-3/2)}}{im\pi} \right]^{s}} - \frac{1}{\left[-2N \cdot \sum_{m=1}^{\infty} \frac{e^{i\pi m(-1/2)} - e^{i\pi m(-3/2)}}{im\pi} \right]^{1-s}} \qquad (6.53A)$$

Then since $e^{i\pi m(-3/2)} = e^{i\pi m(+1/2)}$:

$$= \sum_{N=1}^{\infty} \frac{1}{\left[-2N \cdot \sum_{m=1}^{\infty} \frac{e^{i\pi m(-1/2)} - e^{i\pi m(+1/2)}}{im\pi} \right]^{s}} - \frac{1}{\left[-2N \cdot \sum_{m=1}^{\infty} \frac{e^{i\pi m(-1/2)} - e^{i\pi m(+1/2)}}{im\pi} \right]^{1-s}} \qquad (6.53B)$$

$$= \sum_{N=1}^{\infty} \frac{1}{\left[2N \cdot \sum_{m=1}^{\infty} \frac{e^{i\pi m(+1/2)} - e^{i\pi m(-1/2)}}{im\pi} \right]^{s}} - \frac{1}{\left[2N \cdot \sum_{m=1}^{\infty} \frac{e^{i\pi m(+1/2)} - e^{i\pi m(-1/2)}}{im\pi} \right]^{1-s}} \qquad (6.53C)$$

$$= \sum_{N=1}^{\infty} \frac{1}{\left[N \cdot \sum_{m=1}^{\infty} \frac{2(e^{i\pi m(+1/2)} - e^{i\pi m(-1/2)})}{im\pi} \right]^{s}} - \frac{1}{\left[N \cdot \sum_{m=1}^{\infty} \frac{2(e^{i\pi m(+1/2)} - e^{i\pi m(-1/2)})}{im\pi} \right]^{1-s}} \qquad (6.53D)$$

As shown in Figure 80A, $\sum_{m=1}^{\infty} \frac{2(e^{i\pi m(+1/2)} - e^{i\pi m(-1/2)})}{im\pi} = 1$. Therefore (6.53D) reduces to

$$\zeta(s)\Big|_{1-s}^{s} = \sum_{N=1}^{\infty} \left[\frac{1}{[N \cdot 1]^{s}} - \frac{1}{[N \cdot 1]^{1-s}} \right] = \sum_{N=1}^{\infty} \left[\frac{1}{N^{s}} - \frac{1}{N^{1-s}} \right] \qquad (6.53E)$$

Therefore we will let $\lambda_N = \left[\frac{1}{N^{s}} - \frac{1}{N^{1-s}} \right]$. (6.54)

$$\frac{1}{2 \cdot i \cdot m} \cdot \left[e^{i \cdot \pi \cdot m \left(\frac{1}{2} \right)} - e^{i \cdot \pi \cdot m \cdot \left(\frac{-1}{2} \right)} \right]$$

1
0
-0.333
0
0.2
0
-0.143
0

$$\frac{(-1)^{n}}{(2n+1)} =$$

1
-0.333
0.2
-0.143
0.111
-0.091
0.077
-0.067

$$\sum_{n=0}^{10000} \frac{(-1)^{n}}{(2n+1)} = 0.785 \qquad \frac{\pi}{4} = 0.785$$

The expression for $\pi/4$ $\sum_{m=1}^{1000} \left[\frac{-1}{2 \cdot i \cdot m} \cdot \left[e^{i \cdot \pi \cdot m \left(\frac{-1}{2} \right)} - e^{i \cdot \pi \cdot m \cdot \left(\frac{1}{2} \right)} \right] \right] = 0.785 = \frac{\pi}{4}$

Can be rearranged to $\sum_{m=1}^{10000} \left[\frac{2}{i \cdot m \cdot \pi} \cdot \left[e^{i \cdot \pi \cdot m \left(\frac{1}{2} \right)} - e^{i \cdot \pi \cdot m \cdot \left(\frac{-1}{2} \right)} \right] \right] = 1.000$

Figure 80A – The Gregory Series for $\pi/4$ can be rearranged into a series expression equal to 1.

This was not a slight of hand, choosing $R = 2N$ which led to the summation equal to 1. In fact the summation results in 1 for any choice of *R* and *N* so long as $N > R$. However the choice of $R = 2N$ leads the Gregory Series: $1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 \ldots$, which is beautiful in its simplicity and essential in our work. See Figure 80A.

Returning to the general case, $\left[-R \cdot \sum_{m=1}^{\infty} \frac{e^{i\pi m(N-R)/R} - e^{i\pi m(-N-R)/R}}{im\pi} \right]^{-1}$ will sum to $\frac{1}{N}$. The m^{th} component in the sum would be $-R \cdot \frac{e^{i\pi m(N-R)/R} - e^{i\pi m(-N-R)/R}}{im\pi}$.

We can see from Figure 80B that for randomly chosen *N* and *R* most terms in the series are unfamiliar, but the sum is still $\pi / 4$, the same as it is for the simple Gregory Series. If we compare the series terms in Figure 80A and 80B it is apparent that when both are equivalent, using a simple series such as the Gregory Series will make our work much easier.

And because $\sum_{m=1}^{\infty} \frac{-R}{N} \cdot \frac{e^{i\pi m(N-R)/R} - e^{i\pi m(-N-R)/R}}{4im} = \frac{\pi}{4}$, we find $\sum_{m=1}^{\infty} \frac{-R}{N} \cdot \frac{e^{i\pi m(N-R)/R} - e^{i\pi m(-N-R)/R}}{im\pi} = 1$ through algebraically rearrangement.

We see that it is possible to run the formula one way, that is have an simple series sum to an irrational ratio of π (as was in the first paper). Or, in the reverse scenario, we can have a set of irrational values that sum to 1, which we were able to expand to *n* and 1/*n*.

$N := 2 \qquad R := 7$

$$\frac{-R}{N}\cdot\frac{e^{i\cdot\frac{\pi}{R}\cdot m(N-R)} - e^{-i\cdot\frac{\pi}{R}\cdot m\cdot(N-R)}}{4\cdot i\cdot m}$$

1.368
-0.853
0.253
0.190
-0.341
0.228
0.000
-0.171

$$\sum_{m=1}^{10000}\left[\frac{-R}{N}\cdot\frac{e^{i\cdot\frac{\pi}{R}\cdot m(N-R)} - e^{-i\cdot\frac{\pi}{R}\cdot m\cdot(N-R)}}{4\cdot i\cdot m}\right] = 0.785$$

$$\frac{\pi}{4} = 0.785$$

$N := 4 \qquad R := 11$

$$\frac{-R}{N}\cdot\frac{e^{i\cdot\frac{\pi}{R}\cdot m(N-R)} - e^{-i\cdot\frac{\pi}{R}\cdot m\cdot(N-R)}}{4\cdot i\cdot m}$$

1.251
-0.520
-0.129
0.340
-0.149
-0.124
0.194
-0.048

$$\sum_{m=1}^{10000}\left[\frac{-R}{N}\cdot\frac{e^{i\cdot\frac{\pi}{R}\cdot m(N-R)} - e^{-i\cdot\frac{\pi}{R}\cdot m\cdot(N-R)}}{4\cdot i\cdot m}\right] = 0.785$$

Figure 80B – The summation is always $\frac{\pi}{4}$ for all $N > R$, but the series is not as pretty as the Gregory Series.

We now return to (6.51): $\zeta(s)\Big|_{1-s}^{s} = \sum_{n=1}^{\infty}\frac{1}{n^s}\Big|_{1-s}^{s} =$

$$\sum_{N=1}^{\infty}\frac{1}{\left[-\int_{-N}^{+N}-\sum_{m=1}^{\infty}(e^{i\pi m(N-R)/R})\cdot dN\right]^s}\Bigg|_{1-s}^{s} = \sum_{N=1}^{\infty}\frac{1}{\left[-R\cdot\sum_{m=1}^{\infty}\frac{e^{i\pi m(N-R)/R}-e^{i\pi m(-N-R)/R}}{im\pi}\right]^s}\Bigg|_{1-s}^{s} \qquad (6.55)$$

Recognizing $-R\cdot\sum_{m=1}^{\infty}\frac{e^{i\pi m(N-R)/R}-e^{i\pi m(-N-R)/R}}{im\pi} = \frac{1}{N}$ as the Range Transformation we rewrite (6.55):

$$\sum_{N=1}^{\infty}\frac{1}{\left[-\int_{-N}^{+N}\sum_{m=1}^{\infty}(e^{i\pi m(N-R)/R})\cdot dN\right]^s}\Bigg|_{1-s}^{s} = \sum_{N=1}^{\infty}\frac{1}{[N]^s}\Big|_{1-s}^{s} = \sum_{N=1}^{\infty}\left[\frac{1}{N^s}-\frac{1}{N^{1-s}}\right] \qquad (6.54)$$

And λ_N can be seen as the set of eigenvalues which satisfy the differential (integral) equation:

$$\frac{1}{\left[-\int_{-N}^{+N}\sum_{m=1}^{\infty}(e^{i\pi m(N-R)/R})\cdot dN\right]^{s}}\Bigg|_{1-s}^{s} = \lambda_N \tag{6.55}$$

It is apparent from the way the equations were formed that (6.54) is true not only for the sum over *N*, but also the individual *N* as well.

And since $$\zeta(s)\Big|_{1-s}^{s} = \sum_{n=1}^{\infty}\frac{1}{n^s}\Big|_{1-s}^{s} = \sum_{N=1}^{\infty}\frac{1}{\left[-\int_{-N}^{+N}\sum_{m=1}^{\infty}(e^{i\pi m(N-R)/R})\cdot dN\right]^{s}}\Bigg|_{1-s}^{s} = \sum_{N=1}^{\infty}\lambda_N \tag{6.56}$$

And therefore $$\lambda_N = \frac{1}{N^s}-\frac{1}{N^{1-s}} = \frac{1}{n^s}-\frac{1}{n^{1-s}} \tag{6.57}$$

We can rearrange this to obtain $$\sum_{N=1}^{\infty}\lambda_N = \sum_{n=1}^{\infty}\frac{1}{n^s}\Big|_{1-s}^{s} = \zeta(s)\Big|_{1-s}^{s} \tag{6.58}$$

And the sum of the eigenvalues equals the zeta function evaluated over the action. The eigenvalues are the coefficients which solve the GTE.

Therefore we have a differential equation of the general form:

$$\lambda_N = \frac{1}{\int F(N)\cdot dN}\Bigg|_{1-s}^{s} \tag{6.59A}$$

With $$F(N) = \sum_{m=1}^{\infty}(e^{i\pi m(N-R)/R}) \tag{6.59B}$$

And $$\frac{1}{\int F(N)\cdot dN}\Big|_{1-s}^{s} = \frac{1}{\int_{-N}^{+N}\sum_{m=1}^{\infty}(e^{i\pi m(N-R)/R})\cdot dN}\Big|_{1-s}^{s} = \frac{1}{\left[-R\sum_{m=1}^{\infty}\frac{(e^{i\pi m(N-R)/R})}{im\pi}\Big|_{-N}^{+N}\right]^{s}}\Big|_{1-s}^{s} =$$

$$\frac{1}{[N\cdot 1]^{s}}\Big|_{1-s}^{s} = \left[\frac{1}{N^{s}}-\frac{1}{N^{1-s}}\right] \qquad (6.59C)$$

So ` $$\frac{1}{\int F(N)\cdot dN}\Big|_{1-s}^{s} = \lambda_N \qquad (6.59D)$$

With $$\lambda_N = \left[\frac{1}{N^{s}}-\frac{1}{N^{1-s}}\right] \qquad (6.59E)$$

To finish up, in Section 4 we separated $\frac{1}{n^{s}}\Big|_{1-s}^{s}$ into real and imaginary component equations which we were able to graph and model as spinners. In this section we found a graphically equivalent function made from the summation of component waves. The amplitude coefficient for each component wave can be represented as the components of an infinite series. The relationship of these three players, π, n, and 1, deserve our attention and will be addressed next.

VII Key Transformation Relations and the π Series

To diverge a bit, these series generations are really variations paralleling the earlier work on the Gregory Series for $\pi/4$. Figures 81A thru C show how the Range Transformation, The Unity Transformation, and The π Transformation are simply algebraic manipulations of one another. And Figure 81D shows that the Gregory Series for $\pi/4$ is just a specific case of The π Transformation. With knowledge of the more general The π Transformation we are able to derive other series similar to the Gregory Series. Figure 81E calculates the series for $\pi/3$, $\pi/4$, $\pi/5$, $\pi/7$, and $\pi/11$.

$$X := 0.5$$

$$-\sum_{m=1}^{10000} \frac{e^{i\cdot\pi\cdot m(X-1)} - e^{i\cdot\pi\cdot m\cdot(-X-1)}}{i\cdot m\cdot\pi} = 0$$

A. The Linear Unit Transformation
is equivalent to X over the range -1 to +1

$$X := 5 \qquad R := 12$$

$$-R\cdot\sum_{m=1}^{10000} \frac{e^{i\cdot\frac{\pi}{R}\cdot m(X-R)} - e^{i\cdot\frac{\pi}{R}\cdot m\cdot(-X-R)}}{i\cdot m\cdot\pi} = 5.000$$

B. The Range Transformation
is equivalent to X. This transforms any number X

$$\frac{-R}{X}\cdot\sum_{m=1}^{10000} \frac{e^{i\cdot\pi\cdot m\frac{(X-R)}{R}} - e^{i\cdot\pi\cdot m\cdot\frac{(-X-R)}{R}}}{i\cdot m\cdot\pi} = 1$$

C. The Unity Transformation
by dividing the Range Transformation by X

$$\frac{-R}{X}\cdot\sum_{m=1}^{10000} \frac{e^{i\cdot\frac{\pi}{R}\cdot m(X-R)} - e^{i\cdot\frac{\pi}{R}\cdot m\cdot(-X-R)}}{i\cdot m} = 3.1415$$

D. The π Transformation
by multiplying the Unity Transformation by π.
π cancels from the denominator and
enters the numerator of the right side.
Many series for π can be generated from this.

$$X := 1 \qquad R := 2$$

$$\frac{-R}{X}\cdot\frac{e^{i\cdot\frac{\pi}{R}\cdot m(X-R)} - e^{i\cdot\frac{\pi}{R}\cdot m\cdot(-X-R)}}{4\cdot i\cdot m}$$

1
0
-0.333
0
0.2
0
-0.143
0

$$\frac{(-1)^n}{(2n+1)} =$$

1
-0.333
0.2
-0.143
0.111
-0.091

E. The Gregory Series
is a particular case of the π Transformation
where X=1 and R=2 with both sides divided by4

$$\sum_{n=0}^{10000} \frac{(-1)^n}{(2n+1)} = 0.785 \qquad \frac{\pi}{4} = 0.785$$

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} = \frac{\pi}{4}}$$

$X := 1 \qquad R := 2$

$$\frac{-R}{X} \cdot \frac{e^{i\cdot\frac{\pi}{R}\cdot m(X-R)} - e^{i\cdot\frac{\pi}{R}\cdot m\cdot(-X-R)}}{3\cdot i\cdot m}$$

1.333
0
-0.444
0
0.267
0
-0.19
0

$$\frac{4\cdot(-1)^n}{6n+3} =$$

1.333
-0.444
0.267
-0.19
0.148
-0.121

F. Other π Series

can be generated with this procedure

$$\sum_{n=0}^{10000} \frac{4(-1)^n}{(6n+3)} = 1.047 \qquad \frac{\pi}{3} = 1.047$$

$$\sum_{n=0}^{\infty} \frac{4(-1)^n}{(6n+3)} = \frac{\pi}{3}$$

$X := 1 \qquad R := 2$

$$\frac{-R}{X} \cdot \frac{e^{i\cdot\frac{\pi}{R}\cdot m(X-R)} - e^{i\cdot\frac{\pi}{R}\cdot m\cdot(-X-R)}}{5\cdot i\cdot m}$$

0.8
0
-0.267
0
0.16
0
-0.114
0

$$\frac{4\cdot(-1)^n}{10n+5} =$$

0.8
-0.267
0.16
-0.114
0.089
-0.073

$$\sum_{n=0}^{10000} \frac{4(-1)^n}{(10n+5)} = 0.628 \qquad \frac{\pi}{5} = 0.628$$

$$\sum_{n=0}^{\infty} \frac{4(-1)^n}{(10n+5)} = \frac{\pi}{5}$$

$X := 1 \qquad R := 2$

$$\frac{-R}{X} \cdot \frac{e^{i\cdot\frac{\pi}{R}\cdot m(X-R)} - e^{i\cdot\frac{\pi}{R}\cdot m\cdot(-X-R)}}{7\cdot i\cdot m}$$

0.571
0
-0.19
0
0.114
0
-0.082
0

$$\frac{4\cdot(-1)^n}{14n+7} =$$

0.571
-0.19
0.114
-0.082
0.063
-0.052

$$\sum_{n=0}^{10000} \frac{4(-1)^n}{(14n+7)} = 0.449 \qquad \frac{\pi}{7} = 0.449$$

$$\sum_{n=0}^{\infty} \frac{4(-1)^n}{(14n+7)} = \frac{\pi}{7}$$

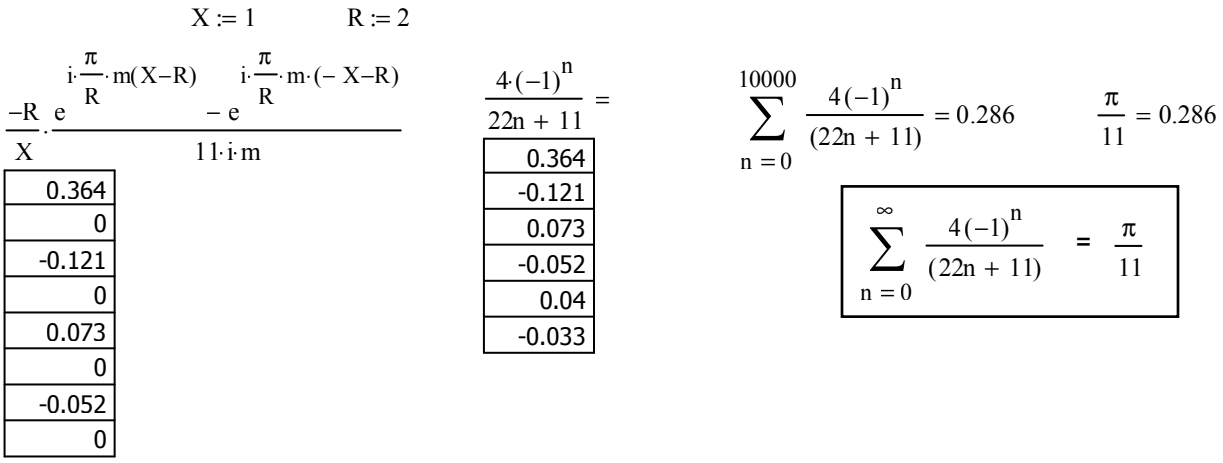


Figure 81 – The Linear Unit Transformation is expanded over range *R* into the Range Transformation. The Range, Unity & π Transformations are algebraic manipulations of one another and can be used to develop the Gregory Series for $\pi/4$, or a unique series for any ratio that is desired.

The series for π/k shown in the Figure 79 calculations are

$$\frac{\pi}{3}=\sum_{n=0}^{\infty}\frac{4(-1)^n}{6n+3},\quad \frac{\pi}{5}=\sum_{n=0}^{\infty}\frac{4(-1)^n}{10n+5},\quad \frac{\pi}{7}=\sum_{n=0}^{\infty}\frac{4(-1)^n}{14n+7},\ \frac{\pi}{11}=\sum_{n=0}^{\infty}\frac{4(-1)^n}{22n+11},$$

as well as the Gregory Series, $\frac{\pi}{4}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=\sum_{n=0}^{\infty}\frac{4(-1)^n}{8n+4}$.

It is apparent that the general relation for transforming any π/k into series is

$$\frac{\pi}{k}=\sum_{n=0}^{\infty}\frac{4(-1)^n}{2kn+k} \tag{6.60}$$

VIII. References & Dedication

References 1 & 2)

"The Riemann Hypothesis" by Enrico Bombieri, page 1

Most if not all of the referenced subsections are taken directly from Bombieri's paper with little to no alteration. The reader should be familiar with Bombieri's work and recognize that we rely on his statement of the problem in these subsections, to be a premise for our work. These subsections are not original work.

Dedication

For my teachers, past and present