POLYLOG SERIES

USING THE METHOD OF INTEGRATION TO DETERMINE HIGHER ORDER MAPPING FUNCTIONS FROM THE POLYLOGARITHM OF THE EULER COMPLEX EXPONENTIAL FUNCTION

Here we use integration of the polylog $\mathrm{Li}_p(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^p}$ to determine the higher order Harmonic Space mapping functions $F_p(\Theta)$. The Euler Complex Exponential Function $e^{i\cdot\Theta}$ is used as the argument z in the polylog.

So the Polylog of the Euler Complex Exponential Function is used for the Harmonic Space mapping functions: $F_p(\Theta) = \sum_{n=1}^{\infty} \frac{e^{in\cdot\Theta}}{n^p}$ with $F_1(\Theta) = \sum_{n=1}^{\infty} \frac{e^{in\cdot\Theta}}{n}$ for the first order.

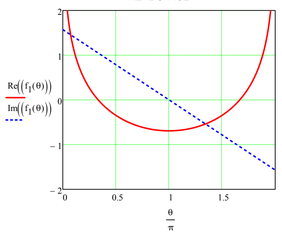
The polylog has an equivalent definition $\mathrm{Li}_1(z) = -\ln(1-z)$ for p = 1, and $\mathrm{Li}_{p+1}(z) = \int_0^z \frac{\mathrm{Li}_p(1)}{t}\,dt$ for all other p. This is used to determine the real space mapping functions $f_p(\theta)$ and is a transform relation in itself.

The first order polylog $\mathrm{Li}_1(z)$ at $z = e^{i\cdot\theta}$ is used for the Real Space mapping function so $f_1(\theta) = -\ln(1-e^{i\cdot\theta})$.

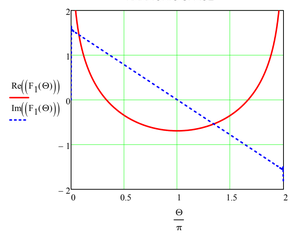
FIRST ORDER

$f_1(\theta) = -\ln(1-e^{i\cdot\theta})$ $F_1(\Theta) = \sum_{n=1}^{\infty} \frac{e^{in\cdot\Theta}}{n}$

REAL SPACE



HARMONIC SPACE



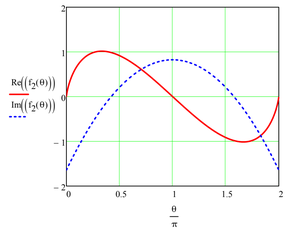
$f_1(0.001) = 6.908 + 1.570i$ $f_1\left(\frac{\pi}{2}\right) = -0.347 + 0.785i$ $f_1(\pi) = -0.693$

$F_1(0) = 7.485$ $F_1\left(\frac{\pi}{2}\right) = -0.346 + 0.785i$ $F_1(\pi) = -0.693$

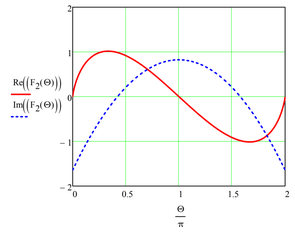
SECOND ORDER

$f_2(\theta) = \int_0^{\theta} -\ln(1-e^{i\cdot\theta})\,d\theta - i\cdot\frac{\pi^2}{6}$ $F_2(\Theta) = -i\sum_{n=1}^{\infty} \frac{e^{i\cdot n\cdot\Theta}}{n^2}$

REAL SPACE



HARMONIC SPACE



$f_2(0) = -1.645i$ $f_2\left(\frac{\pi}{2}\right) = 0.916 + 0.206i$ $f_2(\pi) = 0.822i$

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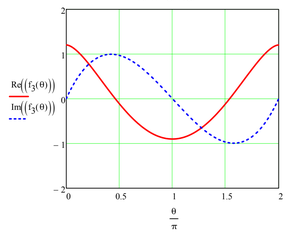
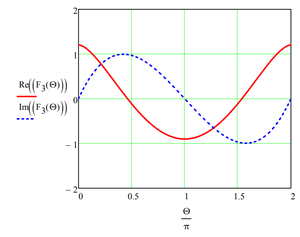
THIRD ORDER

$f_3(\theta) = \int_0^{\theta}\int_0^{\theta} \ln(1-e^{i\cdot\theta})\,d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3$ $F_3(\Theta) = \sum_{n=1}^{\infty} \frac{e^{i\cdot n\cdot\Theta}}{n^3}$

REAL SPACE



HARMONIC SPACE



$f_3(0) = 1.202$ $f_3\left(\frac{\pi}{2}\right) = -0.113 + 0.969i$ $f_3(\pi) = -0.902$

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FIRST ORDER

Use as first order functions $f_1(\theta) = -\ln(1-e^{i\cdot\theta})$ for Real Space and $F_1(\Theta) = \sum_{n=1}^{\infty} \frac{e^{in\cdot\Theta}}{n}$ for Harmonic Space which are shown to be graphically equivalent

Real Space

$f_1(\theta) = -\ln(1-e^{i\cdot\theta})$

Harmonic Space

$F_1(\Theta) = \sum_{n=1}^{\infty} \frac{e^{in\cdot\Theta}}{n}$

Then the first order mapping relationship is $-\ln(1-e^{i\cdot\theta}) = \sum_{n=1}^{\infty} \frac{e^{in\cdot\Theta}}{n}$

Real Space Harmonic Space

Evaluate at π

Real Space

$-\ln(1-e^{i\cdot\pi}) = -\ln(2) = -0.693$

Harmonic Space

$\frac{e^{in\cdot\pi}}{n} = \begin{pmatrix} -1.000 \\ 0.500 \\ -0.333 \\ 0.250 \\ -0.200 \\ 0.167 \\ -0.143 \\ 0.125 \end{pmatrix} = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \mathrm{Li}_1(-1)$

So $-\ln(2) = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \mathrm{Li}_1(-1)$

Therefore $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln(2) = -\mathrm{Li}_1(-1)}$

Evaluate at π/2

$-\ln\left(1-e^{-i\cdot\frac{\pi}{2}}\right) = -\ln(1-i) = -0.347 + 0.785i$

$\frac{e^{in\cdot\frac{\pi}{2}}}{n} = \frac{i^n}{n} = \begin{pmatrix} 1.000i \\ -0.500 \\ -0.333i \\ 0.250 \\ 0.200i \\ -0.167 \\ -0.143i \\ 0.125 \end{pmatrix} = i\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \mathrm{Li}_1(i)$

So $-\ln(1-i) = i\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \mathrm{Li}_1(i)$

[1]But $-\ln(1-i) = i\cdot\frac{\pi}{4} - \frac{1}{2}\ln(2)$

So $i\cdot\frac{\pi}{4} - \frac{1}{2}\ln(2) = i\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

[2]Only the Imaginary Component is needed, the Real is redundant:

Therefore $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4} = \frac{\ln(1+i)-\ln(1-i)}{2\cdot i} = \frac{\mathrm{Li}_1(i)-\mathrm{Li}_1(-i)}{2\cdot i}}$

[1]But $-\ln(1-i) = i\cdot\frac{\pi}{4} - \frac{1}{2}\ln(2)$

Since $\ln(a+b\cdot i) = \ln\left(\sqrt{a^2+b^2}\right) + i\cdot\mathrm{atan}\left(\frac{b}{a}\right)$

So $-\ln(1-i) = \ln(\sqrt{2}) + i\cdot\mathrm{atan}(-1)$

$= \frac{1}{2}\cdot\ln(2) - i\cdot\frac{\pi}{4}$

[2]Note how the complex components can be separated algebraically:

$\frac{\ln(1+i)-\ln(1-i)}{2\cdot i} = \frac{\pi}{4}$

$\frac{\ln(1+i)+\ln(1-i)}{2} = \frac{1}{2}\cdot\ln(2)$

with the general rule:

$\mathrm{Re}\left(-i\sum_{n=1}^{\infty} \frac{e^{i\cdot n\cdot\Theta}}{n^p}\right) \frac{\mathrm{Li}_p(i)-\mathrm{Li}_p(-i)}{2\cdot i}$ $\mathrm{Im}\left(-i\sum_{n=1}^{\infty} \frac{e^{i\cdot n\cdot\Theta}}{n^p}\right) \frac{\mathrm{Li}_p(i)+\mathrm{Li}_p(-i)}{2}$

SECOND ORDER

Integrate the first order Real Space and Harmonic Space mapping functions to obtain the second order mapping functions.
Because the first order functions are graphically equivalent, the second order functions will be as well:

Real Space

$\int f_1(\theta)\,d\theta = \int -\ln(1-e^{i\cdot\theta})\,d\theta = \int_0^{\theta} -\ln(1-e^{i\cdot\theta})\,d\theta - i\cdot C_2 \quad C_2 = \frac{\pi^2}{6}$

(C is determined by evaluation at Θ = 0)

Harmonic Space

$\int F_1(\Theta)\,d\Theta = \int \sum_{n=1}^{\infty} \frac{e^{i\cdot n\cdot\Theta}}{n}\,d\Theta = -i\sum_{n=1}^{1000} \frac{e^{i\cdot n\cdot\Theta}}{n^2}$

Then the second order mapping relationship is $\int_0^{\theta} -\ln(1-e^{i\cdot\theta})\,d\theta - i\cdot\frac{\pi^2}{6} = -i\sum_{n=1}^{\infty} \frac{e^{i\cdot n\cdot\Theta}}{n^2}$

Real Space Harmonic Space

Evaluate at 0

Real Space

$\int_0^{0} -\ln(1-e^{i\cdot\theta})\,d\theta - i\cdot\frac{\pi^2}{6} = -1.645i$

Harmonic Space

$\frac{-i\cdot e^{i\cdot n\cdot 0}}{n^2} = \begin{pmatrix} -1.000i \\ -0.250i \\ -0.111i \\ -0.063i \\ 0.040 \\ 0.028i \\ -0.020 \\ -0.016i \end{pmatrix} = -i\sum_{n=1}^{\infty} \frac{1}{n^2} = -i\mathrm{Li}_2(1)$

So $\int_0^{0} -\ln(1-e^{i\cdot\theta})\,d\theta - i\cdot\frac{\pi^2}{6} = -i\sum_{n=1}^{\infty} \frac{1}{n^2} = -i\mathrm{Li}_2(1)$

But $\int_0^{0} -\ln(1-e^{i\cdot\theta})\,d\theta = 0$

Therefore $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = \mathrm{Li}_2(1)}$

And $\boxed{\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8} = \frac{3}{4}\mathrm{Li}_2(1) = 1.234..}$

Evaluate at π/2

$\int_0^{\frac{\pi}{2}} -\ln(1-e^{i\cdot\theta})\,d\theta - i\cdot\frac{\pi^2}{6} = 0.916 + 0.206i$

$\frac{-i\cdot e^{i\cdot n\cdot\frac{\pi}{2}}}{n^2} = \begin{pmatrix} 1.000 \\ 0.250i \\ -0.111 \\ -0.063i \\ 0.040 \\ 0.028i \\ -0.020 \\ -0.016i \end{pmatrix} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} + i\cdot\frac{1}{4}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = -i\mathrm{Li}_2(i)$

So $\int_0^{\frac{\pi}{2}} -\ln(1-e^{i\cdot\theta})\,d\theta - i\cdot\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} + i\cdot\frac{1}{4}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = -i\mathrm{Li}_2(i)$

[3]Only the Real Component is needed, the Imaginary is redundant:

Re $\int_0^{\frac{\pi}{2}} -\ln(1-e^{i\cdot\theta})\,d\theta - i\cdot\frac{\pi^2}{6} = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\ln(1-e^{i\cdot\theta})\,d\theta = 0.916$

So $\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\ln(1-e^{i\cdot\theta})\,d\theta = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} = \frac{\mathrm{Li}_2(i)-\mathrm{Li}_2(-i)}{2\cdot i}$

Therefore $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\ln(1-e^{i\cdot\theta})\,d\theta = \frac{\mathrm{Li}_2(i)-\mathrm{Li}_2(-i)}{2\cdot i} = 0.916..}$

Evaluate at π

$\int_0^{\pi} -\ln(1-e^{i\cdot\theta})\,d\theta - i\cdot\frac{\pi^2}{6} = 0.822i$

$\frac{-i\cdot e^{i\cdot n\cdot\pi}}{n^2} = \begin{pmatrix} 1.000i \\ -0.250i \\ 0.111i \\ -0.063i \\ 0.040i \\ -0.028i \\ 0.020i \\ -0.016i \end{pmatrix} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = -i\mathrm{Li}_2(-1)$

Sc $\int_0^{\pi} -\ln(1-e^{i\cdot\theta})\,d\theta - i\cdot\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = -i\mathrm{Li}_2(-1)$

But $\int_0^{\pi} -\ln(1-e^{i\cdot\theta})\,d\theta = i\cdot\frac{\pi^2}{4}$

Therefore $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12} = -\mathrm{Li}_2(-1) = 0.822..}$

THIRD ORDER

Integrate the second order functions to obtain the third order:

Real Space

$\int f_2(\theta)\,d\theta = \int \int_0^{\theta} -\ln(1-e^{i\cdot\theta})\,d\theta - i\cdot\frac{\pi^2}{6}\,d\theta = -\int_0^{\theta}\int_0^{\theta} \ln(1-e^{i\cdot\theta})\,d\theta + i\cdot\frac{\pi^2}{6}\,d\theta - C_3$ *with $C_3 = -\frac{4}{7}\int_0^{\pi}\int_0^{\theta} \ln(1-e^{i\cdot\theta})\,d\theta + i\cdot\frac{\pi^2}{6}\,d\theta = -\frac{2}{7}\int_0^{\pi}\int_{-\theta}^{\theta} \ln(1-e^{i\cdot\theta})\,d\theta\,d\theta = \sum_n \frac{1}{n^3} = \mathrm{Li}_3(1)$

Harmonic Space

$\int F_2(\Theta)\,d\Theta = \int -i\sum_{n=1}^{\infty} \frac{e^{i\cdot n\cdot\Theta}}{n^2}\,d\Theta = -\sum_{n=1}^{\infty} \frac{e^{i\cdot n\cdot\Theta}}{n^3}$

Both the Real & Harmonic Mapping Functions can be negated for simplicity.

(C is determined by evaluation at Θ = 0)

Then the third order mapping relationship is $\int_0^{\theta}\int_0^{\theta} \ln(1-e^{i\cdot\theta})\,d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3 = \sum_{n=1}^{\infty} \frac{e^{i\cdot n\cdot\Theta}}{n^3}$

Real Space Harmonic Space

Evaluate at 0

Real Space

$\int_0^{0}\int_0^{0} \ln(1-e^{i\cdot\theta})\,d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3 = 1.202$

Harmonic Space

$\frac{e^{i\cdot n\cdot 0}}{n^3} = \begin{pmatrix} 1.000 \\ 0.125 \\ 0.037 \\ 0.016 \\ 0.008 \\ 0.005 \\ 0.003 \\ 0.002 \end{pmatrix} = \sum_{n=1}^{\infty} \frac{1}{n^3} = \mathrm{Li}_3(1)$

So $\int_0^{0}\int_0^{0} \ln(1-e^{i\cdot\theta})\,d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3 = \sum_{n=1}^{\infty} \frac{1}{n^3} = \mathrm{Li}_3(1)$

But $\int_0^{0}\int_0^{0} \ln(1-e^{i\cdot\theta})\,d\theta + i\cdot\frac{\pi^2}{6}\,d\theta = 0$

Therefore $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{2}{7}\int_0^{\pi}\int_{-\theta}^{\theta} -\ln(1-e^{i\cdot\theta})\,d\theta\,d\theta = C_3 = \mathrm{Li}_3(1) = 1.202..}$

And $\boxed{\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} = \frac{7}{8}\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{7}{8}\cdot C_3 = \frac{7}{8}\mathrm{Li}_3(1) = 1.052..}$

Evaluate at π/2

$\int_0^{\frac{\pi}{2}}\int_0^{\theta} \ln(1-e^{i\cdot\theta})\,d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3 = -0.113 + 0.969i$

$\frac{e^{i\cdot n\cdot\frac{\pi}{2}}}{n^3} = \begin{pmatrix} 1.000i \\ -0.125 \\ -0.037i \\ 0.016 \\ 0.008i \\ -0.005 \\ -0.003i \\ 0.002 \end{pmatrix} = i\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} - \frac{1}{8}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} = \mathrm{Li}_3(i)$

So $\int_0^{\frac{\pi}{2}}\int_0^{\theta} \ln(1-e^{i\cdot\theta})\,d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3 = i\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} - \frac{1}{8}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} = \mathrm{Li}_3(i)$

[5]Only the Imaginary Component is needed, the Real is redundant:

Im $\int_0^{\frac{\pi}{2}}\int_0^{\theta} \ln(1-e^{i\cdot\theta})\,d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3 = -\frac{1}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi} \ln(1-e^{i\cdot\theta})\,d\theta\,d\theta = 0.969i$

So $-\frac{1}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi} \ln(1-e^{i\cdot\theta})\,d\theta\,d\theta = i\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3}$

Therefore $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{32} = -\frac{1}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi} \ln(1-e^{i\cdot\theta})\,d\theta\,d\theta = \frac{\mathrm{Li}_3(i)-\mathrm{Li}_3(-i)}{2\cdot i} = 0.969..}$

Evaluate at π

$\int_0^{\pi}\int_0^{\theta} \ln(1-e^{i\cdot\theta})\,d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3 = -0.902$

$\frac{e^{i\cdot n\cdot\pi}}{n^3} = \begin{pmatrix} -1.000 \\ 0.125 \\ -0.037 \\ 0.016 \\ -0.008 \\ 0.005 \\ -0.003 \\ 0.002 \end{pmatrix} = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} = \mathrm{Li}_3(-1)$

So $\int_0^{\pi}\int_0^{\theta} \ln(1-e^{i\cdot\theta})\,d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3 = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} = \mathrm{Li}_3(-1)$

[6]But $\int_0^{\pi}\int_0^{\theta} \ln(1-e^{i\cdot\theta})\,d\theta + i\cdot\frac{\pi^2}{6}\,d\theta = -\frac{7}{4}C_3 + C_3 = -\frac{3}{4}C_3$

Sc $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} = \frac{3}{14}\int_0^{\pi}\int_{-\theta}^{\theta} -\ln(1-e^{i\cdot\theta})\,d\theta\,d\theta$

Therefore $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} = \frac{3}{4}C_3 = -\mathrm{Li}_3(-1) = 0.902..}$

[4]The integration constant $C_2 = -i\cdot\frac{\pi^2}{6}$ can often be eliminated by using symmetry in the integral bounds:

$C_3 = -\frac{4}{7}\int_0^{\pi}\int_0^{\theta} \ln(1-e^{i\cdot\theta})\,d\theta + i\cdot\frac{\pi^2}{6}\,d\theta = -\frac{2}{7}\int_0^{\pi}\int_{-\theta}^{\theta} \ln(1-e^{i\cdot\theta})\,d\theta\,d\theta = 1.202$

[3]Note how the real portion of the integral can be isolated using symmetry in the bounds of the integral. Recall that subtracting the opposing spinners when evaluating the integral negates the one componentand the other component doubles:

$\int_0^{\frac{\pi}{2}} -\ln(1-e^{i\cdot\theta})\,d\theta - i\cdot\frac{\pi^2}{6} = 0.916 + i\cdot\frac{\pi^2}{48} = 0.916 + 0.206i$

$\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\ln(1-e^{i\cdot\theta})\,d\theta = 0.916$

[5]Note separating the Imaginary Component out by using symmetry in the integral bounds:

Im $-\int_0^{\frac{\pi}{2}}\int_0^{\theta} \ln(1-e^{i\cdot\theta})\,d\theta + i\cdot\frac{\pi^2}{6}\,d\theta - C_3 = 0.113 - 0.969i = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} i\cdot\frac{\pi^2}{6} + \int_0^{\theta} \ln(1-e^{i\cdot\theta})\,d\theta\,d\theta = 0.969i$

and the integration constant C_3 can be removed as done above $= -\frac{1}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi} \ln(1-e^{i\cdot\theta})\,d\theta\,d\theta = 0.969i$

[6]But $\int_0^{\pi}\int_0^{\theta} \ln(1-e^{i\cdot\theta})\,d\theta + i\cdot\frac{\pi^2}{6}\,d\theta = -\frac{7}{4}C_3 + C_3 = -\frac{3}{4}C_3$

since $-\frac{4}{7}\int_0^{\pi}\int_0^{\theta} \ln(1-e^{i\cdot\theta})\,d\theta + i\cdot\frac{\pi^2}{6}\,d\theta = C_3 = \sum_{n=1}^{\infty} \frac{1}{n^3}$

and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} = \frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{3}{4}C_3$

[6]But $\int_0^{\pi}\int_0^{\theta} \ln(1-e^{i\cdot\theta})\,d\theta + i\cdot\frac{\pi^2}{6}\,d\theta = -\frac{7}{4}C_3 + C_3 = -\frac{3}{4}C_3$

since $-\frac{4}{7}\int_0^{\pi}\int_0^{\theta} \ln(1-e^{i\cdot\theta})\,d\theta + i\cdot\frac{\pi^2}{6}\,d\theta = C_3 = \sum_{n=1}^{\infty} \frac{1}{n^3}$

and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} = \frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{3}{4}C_3$