Determining the First Order Harmonic Space Mapping Function by Integration of the Basis Space

To determine by integration the Harmonic Space mapping function $F_1(\Theta) = \sum_{n=1}^{\infty}\left[\frac{2}{n}\cdot\sin[n\cdot(\pi-\Theta)]\right]$

Assume the basis polar wave space to be of the form $\sum_{n=1}^{\infty} e^{i\cdot n\cdot\Theta}$ as it continuously defines a circle.

We integrate to the basis space get the mapping function. For continuity on a circle, with angular symmetry around Θ = 0, we want the boundaries to extend from $\Theta = -\pi$ to $\Theta = \pi$.

Solving these for 0 we obtain $\pi+\Theta$ and $\pi-\Theta$. We use these as lower and upper bounds of the integral, execute, and evaluate:

$$F_1(\Theta) = \int F_0(\Theta)\,d\Theta = \int_{\pi+\Theta}^{\pi-\Theta}\sum_n e^{i\cdot n\cdot\Theta}\,d\Theta = \sum_{n=1}^{\infty}\left[\frac{1}{n\cdot i}\left[e^{i\cdot n\cdot(\pi-\Theta)} - e^{i\cdot n\cdot(\Theta+\pi)}\right]\right]$$

With $e^{i\cdot(\Theta+\pi)} = e^{-i\cdot(\pi-\Theta)}$ this simplifies to

Using $F_1(\Theta) = \sum_{n=1}^{\infty}\left[\frac{1}{n\cdot i}\left[e^{i\cdot n\cdot(\pi-\Theta)} - e^{-i\cdot n\cdot(\pi-\Theta)}\right]\right]$

$e^{i\cdot n\cdot(\pi-\Theta)}$ designates a spinner in the complex plane of radius 1 and frequency n. See below.

The set has n pairs of two opposing spinners $\frac{1}{n\cdot i}\cdot e^{i\cdot n\cdot(\pi-\Theta)}$ and $\frac{1}{n\cdot i}\cdot e^{-i\cdot n\cdot(\pi-\Theta)}$ subtracted.

With $\sin(a) = \frac{e^{ia} - e^{-ia}}{2\cdot i}$ this simplifies to

$$F_1(\Theta) = \sum_{n=1}^{\infty}\left[\frac{2}{n}\cdot\sin[n\cdot(\pi-\Theta)]\right]$$

which is our first order Harmonic Space mapping function.

Note that prior to this new simplification we have expressed this as $\sum_{n=1}^{\infty}\left[\frac{2}{n\cdot i}\cdot\sinh[in\cdot(\pi-\Theta)]\right]$

These sine and sinh functions are algerbaically equivalent. The sine function makes the wave mechanics more familiar and easier to visualize and will be used going forward.

The introduction of the integral method with appropriate boundary conditions for determining the mapping space function is a more rigid approach than previously which relied more on graphical equivalence.

For symmetry, we write the particluar basis space as

$$F_0(\Theta) = \sum_{n=1}^{\infty} -e^{-i\cdot n\cdot(\pi-\Theta)}$$

Then the graphical equivalence relation can be written as

$$F_1(\Theta) = \int_{-\Theta}^{\Theta} F_0(\Theta)\,d\Theta = \int_{-\Theta}^{\Theta}\sum_{n=1}^{\infty} -e^{-i\cdot n\cdot(\pi-\Theta)}\,d\Theta = \sum_{n=1}^{\infty}\left[\frac{2}{n}\cdot\sin[n\cdot(\pi-\Theta)]\right] \quad \text{(Harmonic Space)}$$

which is graphically equal to $f_1(\theta) = \int_0^{\theta} f_0(\theta)\,d\theta = f_1(\theta) = \theta$ (Real Space)

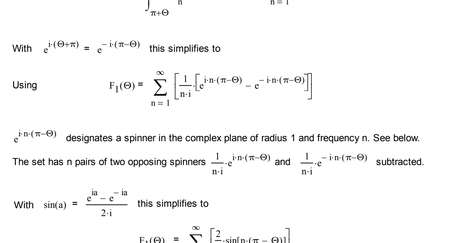
And graphical equivalence at a specific value can be expressed as

$$\theta = \sum_{n=1}^{\infty}\left(\frac{2}{n}\sin(n(\pi-\Theta))\right)$$ where we use the same value for Θ as θ.

Difference of Two Opposing Complex Spinners Generates a Real Function

Examine $\int_{-\Theta}^{\Theta} e^{i\cdot n\cdot\Theta}\,d\Theta = \frac{1}{n\cdot i}\left(e^{i\cdot n\cdot\Theta} - e^{-i\cdot n\cdot\Theta}\right)$

Recall Euler function $e^{i\cdot\Theta}$ generates a unit spinner in the complex plane. Here we have two spinners:



The integral creates two opposing complex spinners that are subtracted, resulting in the imaginary component cancelling out and the real component to double. We end up with a real number (sine function) from the symmetry. The integral needs to be evaluated at specific bounds and in a symmetric manner (-Θ to +Θ).

And because the integral is true for each n it must be good for their sum. The phase shift of Θ by π was not included above for simplicity. Therefore:

$$\int_{\pi+\Theta}^{\pi-\Theta}\sum_n e^{i\cdot n\cdot\Theta}\,d\Theta = \sum_{n=1}^{\infty}\left[\frac{1}{n\cdot i}\left[e^{i\cdot n\cdot(\pi-\Theta)} - e^{-i\cdot n\cdot(\pi-\Theta)}\right]\right] = \sum_{n=1}^{\infty}\left[\frac{2}{n}\cdot\sin[n\cdot(\pi-\Theta)]\right]$$

The measurement process is represented by an integral which when evaluated results in the difference of two complex spinners. This results in the product of 2/n and the sine function which is always a real number. It is sinusoidal and decreases in amplitude by a factor of 1/n.

Connection to Infinite Series

If we evaluate the Harmonic Space mapping function at a specific value each complex wave function will collapse to a real value, and we are left with an infinite sum of real numbers which is always real.

Use $F_1(\Theta) = \sum_{n=1}^{\infty}\left[\frac{2}{n}\cdot\sin[n\cdot(\pi-\Theta)]\right]$ and $F_{1n}(\Theta) = \frac{2}{n}\cdot\sin[n\cdot(\pi-\Theta)]$

For example at $\Theta = \frac{\pi}{2}$ $F_1\left(\frac{\pi}{2}\right) = \sum_{n=1}^{\infty}\left(\frac{2}{n}\sin\left(n\cdot\frac{\pi}{2}\right)\right)$

which is the sum of sine based functions which are always real.

We know $F_1\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$ because of the graphical equivalence between $F_1(\Theta)$ and $f_1(\theta)$, or by statement that the arc length at angle $\theta = \frac{\pi}{2}$ is $\frac{\pi}{2}$.

So $\sum_{n=1}^{\infty}\left(\frac{2}{n}\sin\left(n\cdot\frac{\pi}{2}\right)\right) = \frac{\pi}{2}$

We reduce the numerator to 1 by multiplying both sides of this by $\frac{1}{2}$:

So $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sin\left(n\cdot\frac{\pi}{2}\right)\right) = \frac{\pi}{4}$

Using $\frac{1}{n}\sin\left(n\cdot\frac{\pi}{2}\right)$ to represent each wavefunction in this set we evaluate each member in the series at $\frac{\pi}{2}$

$\frac{1}{n}\cdot\sin\left(n\cdot\frac{\pi}{2}\right) =$

1
0
-0.333
0
0.2
0
-0.143
0

$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\ldots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}$

So $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$

The table shows how evaluating the wave function $F_{1n}(\Theta)$ at π/2 generates the Gregory Series for π.

Surprisingly the wave set that we determine from the permissible wave functions on a circumference at some angle sums to a linear relation equal to the arc length we would measure at that angle. This indicates that when space needs to create this configuration all the permissible harmonics are used.

Graphical Equivalence

Graphical equivalence is provided by the Transformation equation or GTE. Complex valued wave functions are used in a Harmonic Space that is unobserveable. Only specific quantized harmonic modes are permissible due to the geometry. These harmonic modes sum to the observed value in Real Space suggesting that nature pursues all permissible paths simulatneously, and it is our observation that results in the wave wave functions to collapse to an observeable result. Mathematically we see the integral of this infinite set of complex wave functions between the boundaries represent a measurement, and the execution of the integral results in the wave set collapsing into a single real number, formed by the infinite sum of the permissible waves functions evaluated at the boundary values under integration. It is the quantum restrictions in Harmonic Space that fix the value of π and other fundamental constants in a way that they can be generated on demand.

The complex Euler function $e^{i\cdot\Theta}$ can be interpreted as a clock or spinner of radius 1 at position Θ.

The integral of the Basis Space generates a set of n members each with two opposing spinners that are subtracted. This is algebraically equivalent to a set of sine based wave functions:

$$F_1(\Theta) = \int_{-\Theta}^{\Theta}\sum_{n=1}^{\infty} e^{i\cdot n\cdot(\pi-\Theta)}\,d\Theta = \sum_{n=1}^{\infty}\left[\frac{1}{n\cdot i}\left[e^{i\cdot n\cdot(\pi-\Theta)} - e^{-i\cdot n\cdot(\pi-\Theta)}\right]\right] = \sum_{n=1}^{\infty}\left[\frac{2}{n\cdot i}\cdot\sinh[in\cdot(\pi-\Theta)]\right] = \sum_{n=1}^{\infty}\left[\frac{2}{n}\cdot\sin[n\cdot(\pi-\Theta)]\right]$$

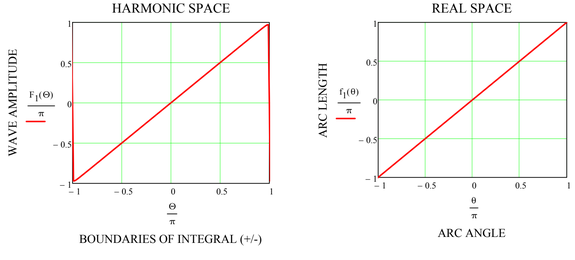
in terms of spinners in terms of sinh in terms of sine

Previously we expressed the first order the Harmonic Space mapping function in terms of sinh(Θ). This has been algebraically simplified to use sin(Θ) which makes the waves mechanics more familiar and easier to visualize.

We regret starting with a sinh based wave functon only to find the simpification to the sine based function that is so familiar.
Our task is like excavating a pyramid one grain at a time. We want to examine these details but there is still much more to uncover.

Below, the first order the Harmonic Space mapping function $F_1(\Theta)$ is plotted on the left. This is an infinite sum of sine based functions.
On the right is the observed measurement for arc length of a circle $f_1(\theta)$ vs arc angle θ. It can be seen that these are graphically equivalent.

$F_1(\Theta) = \sum_{n=1}^{\infty}\left[\frac{2}{n}\cdot\sin[n\cdot(\pi-\Theta)]\right]$ $f_1(\theta) = \theta$



We call this graphical equivalence because the values for the input θ and Θ and the results $f_1(\theta)$ and $F_1(\Theta)$ are numerically equal, but they are not equivalent types of objects. Real Space tracks arc length vs arc angle θ of a circle. Harmonic Space is an infinite sum of wave functions evaluated at the bounds of an integral from -Θ to +Θ. Additionally, it is the sum of the Harmonic Space wave functions that converges to the Real Space function as n, the number of wave functions, approaches infinity. The graphical equivalence becomes absolute only when we consider all of the permisisble waves.

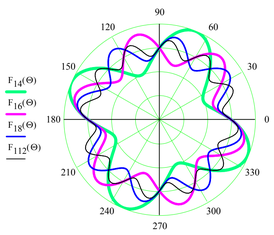
Why Graphical Equivalence Works

Permissible Waves

$$F_1(\Theta) = \sum_{n=1}^{\infty}\left[\frac{2}{n}\cdot\sin[n\cdot(\Theta-\pi)]\right]$$

The first order mapping function is a series of sine waves decreasing in amplitude.

Some of the permissible wave functions are shown on the graph. There are n permissible waves, they are continuous, and have a smoothly changing slope with no kinks.



Exact multiples of the wavelength must fit on the circumference for continuity, or nλ = 2π. These are decreasing in amplitude by a factor of 2/n, which comes from exponent under integration. The slope must change smoothly in order for the mapping functions to maintain continuity under differentiation and integration. Therefore:

$\lambda_n = \frac{2\pi}{n}$ and $\Lambda_n = \frac{2}{n}$

Selection of Basis Space

The underlying basis space is selected to be continuous in the geometry with nλ = 2π, and for the integral of this to do so as well, due to the quantized exponential term.

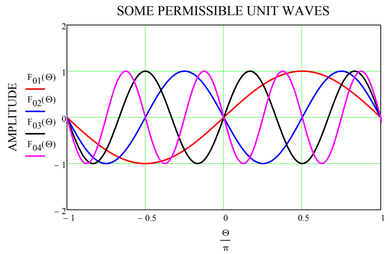
$$F_0(\Theta) = \sum_{n=1}^{\infty} e^{i\cdot n\cdot(\pi-\Theta)}$$

And the offset term π-Θ creates symmetry around 0 with Θ extending from -π to +π.

We can use $\frac{e^{i\cdot n\cdot(\pi-\Theta)} - e^{-i\cdot n\cdot(\pi-\Theta)}}{2\cdot i} = \sin[n\cdot(\pi-\Theta)]$

and $\frac{e^{i\cdot n\cdot(\pi-\Theta)} + e^{-i\cdot n\cdot(\pi-\Theta)}}{2\cdot i} = \cos[n\cdot(\pi-\Theta)]$

if we wish to display the real and imaginary components of the permissible unit waves. We can regard these as the permissible perturbations of space that can exist in this geometry.



The integral of $F_0(\Theta)$ also results in a set of continuous wave functions that fits continuously due to the exponent being transferred under integration. So these will all be permissible wave functions that could exist on a circumference as well, but these have specifc amplitudes 1/n.

This is the set involved in forming or measuring the arc or circle. By design, the integral measures accumulation by calculating the area under a curve with two bounds. Here we integrate the basis space, which is the set of permissible wave functions, over the arc angle.

We see that because the slope of $F_1(\Theta)$ is always 1, the accumulation of the unit wave functions under the integral is going to match the arc angle; ie. at halfway around they will sum to π. In calculating this accumulation, the integral sets the restrictions on the amplitudes and wavelengths of the individual wave functions for this relationship to occur.

To highlight why this important, we can define arc angle θ, geometrically, meaning without numbers. We can determine quarter way around with tools which we define as π/2.
Θ corresponds with θ, and $F_1(\Theta)$ corresponds with $f_1(\theta)$.
When we evaluate $F_1(\pi/2)$ we get an infinte series of rational numbers that sums to π/2.
Thus $F_1(\pi/2)$ gives us a numerical value for the arc length $f_1(\pi/2)$, which was a geometric construction.

The mechanics of integration results in a set of real, sine based functions:

$F_{1n}(\Theta) = \frac{2}{n}\cdot\sin[n\cdot(\pi-\Theta)]$ (invidual wave functions)

$F_1(\Theta) = \int_{-\Theta}^{\Theta}\sum_{n=1}^{\infty} e^{i\cdot n\cdot(\pi-\Theta)}\,d\Theta = \sum_{n=1}^{\infty}\left[\frac{2}{n}\cdot\sin[n\cdot(\pi-\Theta)]\right]$ (mapping equation, sum of infinite set)

The integral results in a 2/n amplitude factor and λ = 2π/n wavelength restriction for each of the n wave functions. At certain key values the restrictions produce a familiar result like the Gregory Series.

The integral represents an accummulation. The arc length at some value Θ includes all of the arc length that has previously been measured. The waveset summation is able to do that because the integral places restrictions on what waves are permissible and how to combine them to make this occur. ***Integrals are often used to find a sum. Here we use them to find the components as well. The integral tells us the wave functions and their amplitude needed to calculate some arc length.***

As the spinners advance angle $\Delta\Theta$ from $e^{i\cdot n\cdot(\pi-\Theta)}$ to $e^{i\cdot n\cdot[\pi-(\Theta+\Delta\Theta)]}$ and $e^{-i\cdot n\cdot(\pi-\Theta)}$ to $e^{-i\cdot n\cdot[\pi-(\Theta+\Delta\Theta)]}$

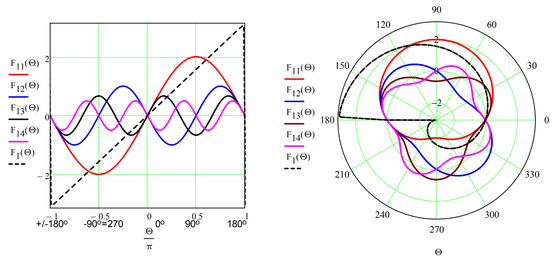
$\Delta F_1(\Theta) = \sum_{n=1}^{\infty}\left(\frac{2}{n}\sin(n\cdot\Delta\Theta)\right)$ which equals $\sum_{n=1}^{\infty}\left[\frac{2}{n}\cdot(n\cdot\Delta\Theta)\right] = 2\cdot\Delta\Theta$ at small $\Delta\Theta$. Each spinner moves $\Delta\Theta$

so the slope is $\frac{2\cdot\Delta\Theta}{2\cdot\Delta\Theta} = 1$ *(Right)*

Waveset is Unsymetrical

$F_{1n}(\Theta) = \frac{2}{n}\sin(n(\pi-\Theta))$ & $F_1(\Theta) = \sum_{n=1}^{\infty}\left(\frac{2}{n}\sin(n(\pi-\Theta))\right)$

individual wave sum



Above, Permissible waves and their sum, first order $F_{1n}(\Theta)$ and their sum $F_1(\Theta)$. The first few wave mapping functions $F_{11}(\Theta)$, $F_{12}(\Theta)$ ...$F_{1n}(\Theta)$ are shown as solid. The dashed line $F_1(\Theta)$ is the sum of this infinite series of waves. The rectangular graph is an "unrolled" view along a circumference running from -π to +π. The wave functions and their slopes are seen to be continuous all around the circle. The wavelengths are quantized and the slopes change smoothly. The infinite series of waves converges to a linear function $F_1(\Theta)$ with a slope of 1 as shown earlier. So at an angular measurement Θ = π, the set of permissible waves sums to the arc length $F_1(\pi) = \pi$. We can see in the polar graph how the lopsided-ness of the waves can sum to a linear trace in the rectangular plot and a spiral trace in the polar plot.

By convention the circumference measurement is zero at Θ = 0 and is negative for negative Θ down to -π/2 and positive for positive Θ up to π/2. And we center our measurement around zero symmetrically. The discontinuity at π/2 in the sum is where the negative and positive ends of the curved ruler meet. The amplitude 2 is determined mathematically.

Zero for the y and r axes are at the circumference, the waves are displacements away from that.
The sum mapping function is the infinite sum of the n wave functions evaluated at that point. It is linear with angle measurement Θ and is the sum of the amplitudes of all the allowable waves, a construction which we identify as a circumference in polar space.

The Slope of the Mapping Function is Always 1

The integration of the Harmonic space function creates n pairs of opposing complex spinners that are subtracted. Symmetry between each of the two spinners causes the complex components to cancel out yielding a real result, a set of sine based waves that decrease in amplitude and increase in frequency. These converge to a single line as n approaches infinity. It extends from -π to +π, has a slope of 1, and passes through the origin. With a slope of 1, and a range of -π to +π, it also maps arc length vs. arc angle. This occurs because the mapping function always has a slope of 1, and is set symmetrically arounnd Θ = 0.

Regarding the slope, if we look at the complex form of the mapping function

$$F_1(\Theta) = \int_{-\Theta}^{\Theta}\sum_{n=1}^{\infty} e^{i\cdot n\cdot(\pi-\Theta)}\,d\Theta = \sum_{n=1}^{\infty}\left[\frac{1}{n\cdot i}\left[e^{i\cdot n\cdot(\pi-\Theta)} - e^{-i\cdot n\cdot(\pi-\Theta)}\right]\right]$$

$e^{i\cdot n\cdot(\pi-\Theta)}$ lies on the unit circle and always has a magnitude of 1. We integrate the sum of this set to get the mapping function.

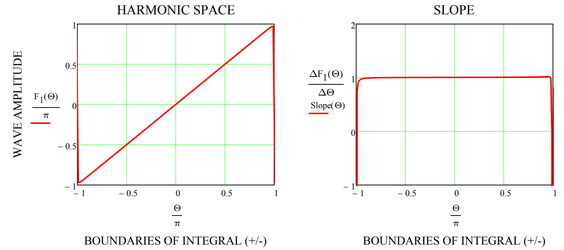
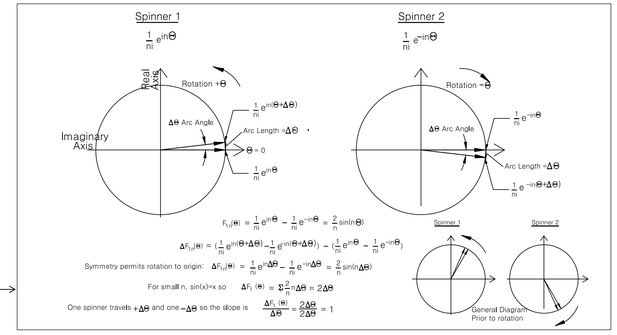
Since it integrates to form the mapping function, then it should be the slope. But it is an infinite series that does not close.

But if we regard this as set as a unit that is later to be portioned into components, the slope would be: $\lim_{k\to\infty}\frac{\sum_{n=1}^{k}\left|e^{i\cdot n\cdot\pi}\right|}{k} = 1$

The waveset does have this slope. Graphically if we look at some small ΔΘ near any value Θ:

$$\Delta F_1(\Theta) = \sum_{n=1}^{\infty}\left[\frac{1}{n\cdot i}\left[e^{i\cdot n\cdot[\pi-(\Theta+\Delta\Theta)]} - e^{i\cdot n\cdot(\pi-\Theta)}\right]\right] - \sum_{n=1}^{\infty}\left[\frac{1}{n\cdot i}\left[e^{-i\cdot n\cdot[\pi-(\Theta+\Delta\Theta)]} - e^{-i\cdot n\cdot(\pi-\Theta)}\right]\right]$$

Then the slope of $F_1(\Theta)$ is:

$$\frac{\Delta F_1(\Theta)}{\Delta\Theta} = \sum_{n=1}^{\infty}\left[\frac{1}{n\cdot i\cdot\Delta\Theta}\left[\left[e^{i\cdot n\cdot[\pi-(\Theta+\Delta\Theta)]} - e^{i\cdot n\cdot(\pi-\Theta)}\right] - \left[e^{-i\cdot n\cdot[\pi-(\Theta+\Delta\Theta)]} - e^{-i\cdot n\cdot(\pi-\Theta)}\right]\right]\right]$$

which is always 1, for all $-\pi < \Theta < \pi$: