Casey's Curves

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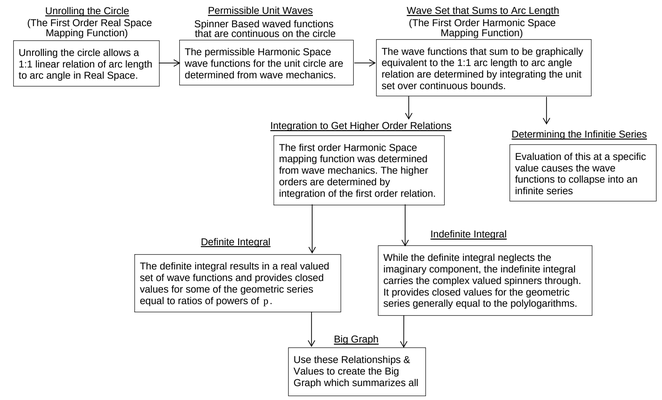
We want to determine closed forms for all of the the power series of the form: $\sum_{n=1}^{\infty}\frac{1}{n^p}$, $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^p}$, $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^p}$ and $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^p}$ for all integers p. The results are seen to follow the general pattern that is shown below for the first five series, with p being the order, or power of the series. With this procedure the series for all higher p can be determined.

We make use of the polylogarithm $Li_p(z)$. For the first order $Li_1(z) = -\ln(1-z)$. For the second order (and up) $Li_2(z) = \int_0^z \frac{Li_1(z)}{z}dz$. In comparison the natural log is $\ln(x) = \int_1^x \frac{1}{x}dx$ which has no higher orders.

Another form of the polylog is $Li_p(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^p}$ which makes it appear to be more fundamental than the natural log because it integrates cleanly in Harmonic Space. $\ln(z) = \sum_{n=1}^{\infty}\frac{(1-z)^n}{n}$ which does not integrate in an aesthetic form. Perhaps we should define the natural log in terms of the polylog!

FLOW CHART



CLOSED FORMS

ORDER	*ALL SERIES*	*ALTERNATING ALL SERIES*	*ODD SERIES*	*ALTERNATING ODD SERIES*
p = 1	$\sum_{n=1}^{\infty}\frac{1}{n} = \infty$ Does Not Converge	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = \ln(2) = 0.693...$	$\sum_{n=1}^{\infty}\frac{1}{2n-1} = \infty$ Does Not Converge	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4} = 0.785...$
p = 2	$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} = 1.645...$	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12} = 0.822...$	$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8} = 1.234...$	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2} = \frac{Li_2(i)-Li_2(-i)}{2\cdot i} = 0.916...$
p = 3	$\sum_{n=1}^{\infty}\frac{1}{n^3} = Li_3(1) = 1.202...$	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3} = -Li_3(-1) = 0.902...$	$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3} = \frac{7}{8}Li_3(1) = 1.052...$	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{32} = 0.969...$
p = 4	$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90} = 1.08232...$	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4} = \frac{7\cdot\pi^2}{720} = 0.947...$	$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96} = 1.015...$	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^4} = \frac{Li_4(i)-Li_4(-i)}{2\cdot i} = 0.989...$
p = 5	$\sum_{n=1}^{\infty}\frac{1}{n^5} = Li_5(1) = 1.037...$	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^5} = -Li_5(-1) = 0.972...$	$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5} = \frac{31}{32}Li_5(1) = 1.004...$	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^5} = \frac{5\cdot\pi^5}{1536} = 0.996...$

Note how π^p alternates with successive rows, in each row it either is in the *alternating odd series*, or in *the other three simultaneously*. These three have an algebraic relationship so defining any one defines all three for that order.

Background

We look at the construction of a curved space from waves. For a wave on the circumference of a circle to be permissible, it must be continuous all around in both position and slope. This restriction limits the wave set to specific wave functions determined by the rules of wave mechanics in a circular geometry. The wavelengths are restricted to quantized solutions in order to be continuous, or nλ = 2π. The slope continuity requirement results in descreasing amplitudes, as calculus ditctates the slope be proportional to 1/n where n is the element number within the wave set, as shown later. A power series can result when the waveset is evaluated at certain loacations, typically having a regular pattern that decreases in amplitude, with sign changes determined by the sine or similar functions evaluated at multiples of π/2. The value of these infinite series will normally involve π, e, or a logarithmic function because the waveset corresponds to the permissible waves that can form a unit circle, and the unit circle is defined by the Euler function $e^{i\Theta}$. Its evaluation at multiples of π/2 frequently comes into play..

Another feature of the permissible waveset is that when summed it converges to the function that corresponds to the observed value for the arc length of the unit circle at angle θ. This set of permissible waves are the only the only way space can construct itself when tasked with forming an arc and their sum equals the arc length. Essentially every time we construct a circumference it is a new phenomens for nature; all possible routes are taken account for and weighted with a simple geometric weighting, and the sum of all of these is what is observed.

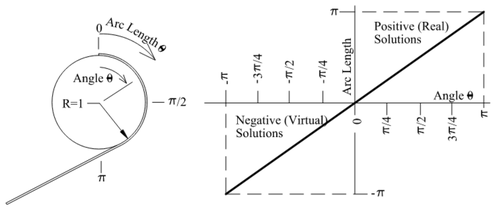
While we can't say for sure that space is constructed this way, we can say that if it is then $\frac{\pi}{4}$ must equal $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}$ which is the well known Gregory Series.

Unrolling the Circle

Unrolling the Circle. With R = 1 the arc angle θ equals the arc length Rθ. We can leave the arc length as unitless, it has whatever unit R does.

The negative values are regarded as the (anti)symmetrical virtual solutions as negative length does not have a physical meaning. These need to be considered in order for the analysis to work.

Virtual solutions arise often in the physics of trajectories. Here we make the analogy that this is like working the trajectory problem backward, starting with both types of solutions and using that to reconstruct the underlying formula.



Constructon of First Order Relationship Based on Wave Mechanics

The construction of a circle is interpreted as the summation of all permissible solutions that exist for a wave-like space to form that geometry. The wave functions forming a circle must be continuous all around and without discontinuity in slope. We visualize the circumference as a continuous string and look at the permissible vibrations. Although our model and graphs depict these waves as transverse they are actually longitudinal as they contribute to the circumference. The waves have no energy, they are unitless, and they lie in a complex plane with imaginary components, which by definition prohibit their detection in our realm. They are the construction of the curved space that objects operate in, and nature creates by summing all possibilities of permissible wave functions that can fit this geometry. Harmonic Space is used to describe the permissible waves used in construction of this non-linear space.

The basis Harmonic Space $F_0(\Theta)$ is the set of permissible wave functions on the unit circle. These are spinners with an amplitude of 1 and frequency f = 2π/n for continuity. In addition to being modeled by spinners, $F_0(\Theta)$ can also be mapped as a set of sinusoidal functions. The integration of this set along the circumference determines the first order Harmonic Space mapping function $F_1(\Theta)$, which is the set of permissible waves summing to the circumference. Because $F_0(\Theta)$ is made up of unit spinners it has an amplitude of 1, so under integration the slope of $F_1(\Theta)$ is 1. With a slope of 1, consider Θ = π , there $F_1(\pi) = \pi$, so the wave functions sum to π at Θ = π. And because $F_1(\Theta)$ is linear, it corresponds to the measured arc length. Like $F_0(\Theta)$, our $F_1(\Theta)$ is also a set of wave functions continuous over the circumference. The complex Euler exponent in $F_0(\Theta)$ causes the same quantization restrictions for continuity to be transfered to $F_1(\Theta)$ under integration. The integral also creates an amplitude factor of 1/n for each of the n wave functions. So $F_1(\Theta)$ is made from permissible waves that are both continuous on the circumference, and add up to its arc length at any angle Θ, and the integration provides a function with the specific amplitudes for them to add up in this way.

Thus the first order Harmonic Space mapping function $F_1(\Theta)$ is an infinite set of n wave functions $F_{1n}(\Theta)$, that when summed is graphically equivalent to the arc length at Θ. The $F_{1n}(\Theta)$ are the set permissible wave functions on a circumference with specific amplitudes determined by integration necessary for the linear relation of $F_1(\Theta)$ with Θ to occur. The wave set converges to the linear relationship as n, the wave number, approaches ∞. Arc length is also the definition our first order Real Space mapping function $f_1(\theta)$. We use a circle with a radius of 1, so $f_1(\theta) = \theta$, or the value of the arc length is equal to the angle. Length units are not displayed but they would match that of the radius. We call the Harmonic Space mapping function $F_1(\Theta)$ and Real Space mapping function $f_1(\theta)$ graphically equivalent since their graphs look the same. This allows us to equate any arc length at θ from $f_1(\theta)$, with a sum of wave functions $F_1(\Theta)$, evaluated at Θ, where θ and Θ correspond 1:1. When evaluated at a specific Θ, the $F_{1n}(\Theta)$ wave functions collapse to n values in an infinite series.

The Real Space mapping function is what we observe, and search for a an explanation as to why it has this value. Harmonic Space is based on Complex Numbers and has imaginary components. This makes it is unobserveable in the space that it defines. In contrast, Real Space contains only Real Numbers and is observeable. But the fundamental constants like π and e are already woven in through wave mechanics of the underlying space, and because of the unobserveable nature of Harmonic Space, reamain elusive to explain. Our premise is that non-linear spaces and their constants are formed by the sum of all permissble waves that can be constructed in space for the geometry. Quantum Mechanics follows a similar reasoning in that the location of a particle can be smeared in space and not clearly defined until a measurement is taken. Our approach is more fundamental. Here it is the underlying space that is quantized due to the geometry placing restrictions on the wave functions necessary to form it.

Nature is constantly tasked to construct a curved space on demand and wave mechanics in Harmonic Space provides a mechanism for it to do that. The mechanism is like a guitar string configured in a circle.

The Measurement Process Reprented by the Integral Results in Collapse of the Complex Valued Wave Function into a Sum of Real Numbers

The mathematical connection between thses two spaces is that a measurement or construction is represented by an integral between two bounds. The integral causes the Harmonic Space function, which is a set of complex waves (at Θ), to collapse into a real number that matches the observations we make for arc length in Real Space (at θ). This is called graphical equivalence. It suggests that space has another orthogonal dimension "i" associated with it which permits wave mechanics to operate to create a curved space. The wave mechanics permit only certain wavefunctions to be permissible, these all exist simultaneously, and the mathematics cause the sum of this construction to be real and observeable.

The graphical equivalence relation is $\int_{-\Theta}^{\Theta} F_0(\Theta)\,d\Theta = \int_0^{\theta} f_0(\theta)\,d\theta$ which integrates to $F_1(\Theta) = f_1(\theta)$

HARMONIC SPACE REAL SPACE HARMONIC SPACE REAL SPACE

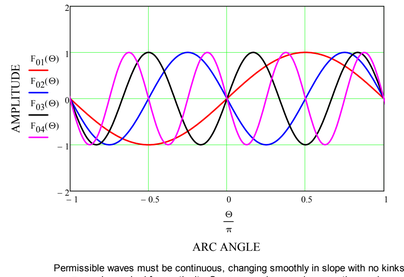
Negative values of Θ are included in integrating the Harmonic Space mapping function, In order for this transformation equation to construction a real result these need to be included. This symmetry is needed for the complex valued wave functions to collapse and sum to a real number. The negative values (-Θ) are regarded as a virtual solutions, similar to what might be encountered in a trajectory or mirror problem. The negative lengths do not have a Real Space counterpart, aren't included in the Real Space integral, and don't contribute to the measure length. This is an asymmetric form of the GTE since we don't include the negative values in Real Space.

For the circle $F_1(\Theta) = \sum_{n=1}^{\infty}\left(\frac{2}{n}\sin(n(\pi-\Theta))\right)$ which is an infinite set of "n" sine waves, one for each n: $F_{1n}(\Theta) = \frac{2}{n}\sin(n(\pi-\Theta))$ in Harmonic Space, and $f_1(\theta) = \theta$ is the arc measurement in Real Space.

And graphical equivalence at a specific value can be expressed as $\theta = \sum_{n=1}^{\infty}\left(\frac{2}{n}\sin(n(\pi-\Theta))\right)$ where we use the same value for Θ as θ.

REAL SPACE HARMONIC SPACE

SOME PERMISSIBLE UNIT WAVES



Permissible waves must be continuous, changing smoothly in slope with no kinks. n·λ = 2π is required for continuity. Some examples are shown on the graph.

since $e^{i\cdot\Theta}$ defines a unit circle the basis space used is $F_0(\Theta) = \sum_{n=1}^{\infty} e^{i\cdot n\cdot\Theta}$.

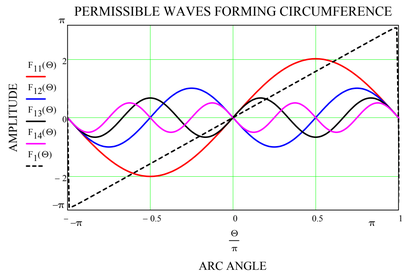
We integrate to get the mapping function, shown later $\int_{-\Theta}^{\Theta}\sum_{n} e^{i\cdot n\cdot\Theta}d\Theta = \sum_{n=1}^{\infty}\left(\frac{2}{n}\sin(n\cdot\Theta)\right)$

Rotations on the circle are permissible, a rotation of π is used for symmetry

The wave functions $F_{1n}(\Theta) = \frac{2}{n}\sin(n(\pi-\Theta))$ are used,

with Harmonic Space mapping function $F_1(\Theta) = \sum_{n=1}^{\infty}\left(\frac{2}{n}\sin(n(\pi-\Theta))\right)$

PERMISSIBLE WAVES FORMING CIRCUMFERENCE



The first few wave functions $F_{1n}(\Theta)$ of the mapping function $F_1(\Theta)$ are shown. They increase in frequency and decreease in amplitude with n.

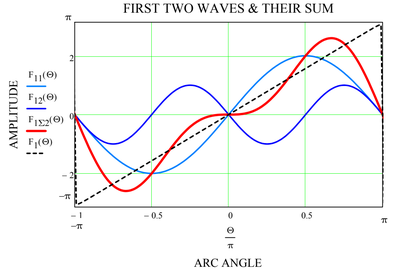
Boundaries run from -π to +π on both axes. The horizontal zero line represents the circumference

n·λ = 2π is required for continuity. The first wave is simplest possible continuous over 2π with one cycle. These increase with n, for n = 2 there are two cycles. The amplitude of each of the n wave decrease by a factor of 1/n. .

We show the waves as up and down (perpendicular) but really are parallel to the circumference contributing to the arc length.

The dashed line represents the sum of all permissible waves and is graphically equivalent to the arc length measurement $f_1(\theta)$.

FIRST TWO WAVES & THEIR SUM



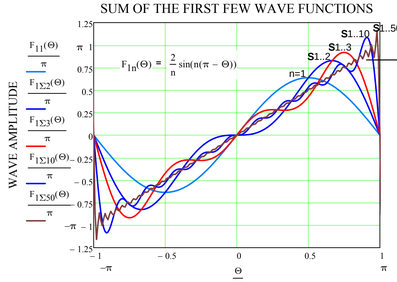
With just the first two waves and their sum the linear pattern can be seen to be forming.

First Order Mapping Functions

$F_{1n}(\Theta) = \frac{2}{n}\sin(n(\pi-\Theta))$	the individual Harmonic Space wave function "n" from the permissible waveset
$F_1(\Theta) = \sum_{n=1}^{\infty}\left(\frac{2}{n}\sin(n(\pi-\Theta))\right)$	Harmonic Space mapping function - the sum the n wave functions
$F_{1\Sigma j}(\Theta) = \sum_{n=1}^{j}\left(\frac{2}{n}\sin(n(\pi-\Theta))\right)$	partial sum used to illustrate progression of convergence to infinite sum
$f_1(\theta) = \theta$	arc length = arc angle in Real Space

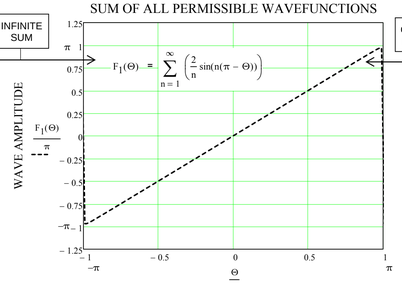
HARMONIC SPACE - SUM OF THE PERMISSIBLE WAVEFUNCTIONS $F_{1n}(\Theta)$

SUM OF THE FIRST FEW WAVE FUNCTIONS



INFINITE SUM

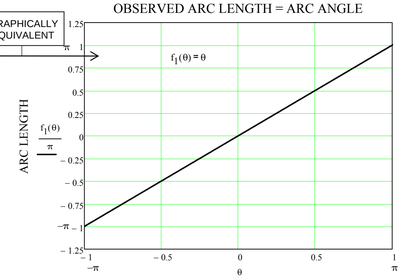
SUM OF ALL PERMISSIBLE WAVEFUNCTIONS



GRAPHICALLY EQUIVALENT

REAL SPACE - $f_1(\theta) = \theta$

OBSERVED ARC LENGTH = ARC ANGLE



The sum of the permissible wave functions for a circular geometry in Harmonic Space (left) are seen to converge (center) to a linear relationship that is graphically equivalent to what we observe in Real Space (right) for the arc length measurement. The first wave function is shown along with the sum of the first two, three, ten & fifty wavefunctions (left), and their sum as n appoaches ∞ (center). The Harmonic Space plots the wave amplitude vs. the boundaries of the integral used in their calculation, with symmetric upper and lower bounds. So 0.5π on the graph represents bounds -π/2 and +π/2 in the calculating integral that detrmines $F_1(\pi/2)$. The integration and boundaries of the basis space $F_0(\Theta)$ to determine the first order map $F_1(\Theta)$ is of the form: $\int_{-\Theta}^{\Theta} F_0(\Theta)\,d\Theta = F_1(\Theta)$

In Real Space we plot arc length versus arc angle. We use a radius of 1, so $f_1(\theta) = \theta$, or the value of the arc length is equal to the arc angle. The integration of the basis space $f_0(\theta)$ to determine the first order map $f_1(\theta)$ is of the form $\int_0^{\theta} f_0(\theta)\,d\theta = f_1(\theta)$ (or $\frac{1}{2}\int_{-\theta}^{\theta} f_0(\theta)\,d\theta$).

This is consistent with the GTE relationship $\int_{-\Theta}^{\Theta} F_0(\Theta)\,d\Theta = \int_0^{\theta} f_0(\theta)\,d\theta$ discussed previously.

HARMONIC SPACE REAL SPACE