# Casey's Curves

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X

continued

**Sum of Products of Sinh Waves**

3. In the third example we map $y = \theta^3$. We can create a Harmonic Space map that is graphically equivalent to this from the product of three sinh waves series. This is similar to the previous example.

$$\theta^3 = \sum_{n=1}^{\infty}\left[32\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\right]\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\Theta}{4}\right)\right]\right]^2\right] \quad \text{(Transform)}$$

(Real) (Harmonic)

If we display the n wave functions before summing we will get the coefficients of an equivalent infinite series.

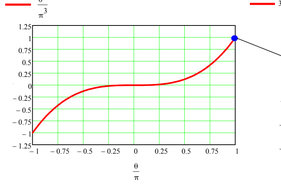
$$\lambda_n = 32\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\right]\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\Theta}{4}\right)\right]\right]^2$$

If we evaluate the Transform above at $\theta = \pi$ and $\Theta = \pi$

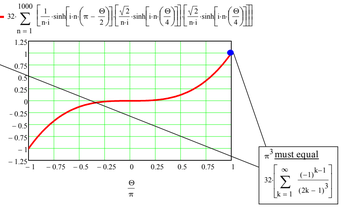
$$\pi^3 = \sum_{n=1}^{\infty}\left[32\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right]\right]\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]^2\right] \quad \text{(Transform)}$$

(Real) (Harmonic)

Real Space




Harmonic Space




$\pi^3$ must equal $32\left[\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^3}\right]$

At π/2 the sum of sinh waves generates the odd series alternating in sign. As we might expect the product of three such terms results in the odd series cubed with the terms alternating in sign.

The table shows $32\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right]\right]\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{2}\right)\right]\right]^2$ equals $32\left(1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{3^3}+\dots\right)$

So $\pi^3 = 32\left(1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\dots\right)$ or $\frac{\pi^3}{32} = \left(1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\dots\right) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^3}$

| 1/2 x Difference of Two Spinners $\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right] =$ | 1/2 x Difference of Two Spinners $\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right] =$ | Square of Previous Column (Product of 2 Diffs) $\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]^2 =$ | 32x Product of These Three Differences $\lambda_n = \frac{32}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right]\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]^2$ | 1/32 of This Product equals the simple series $\frac{1}{32}\lambda_n$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 32 | 1 |
| 0 | 0.707 | 0.5 | 0 | 0 |
| -0.333 | 0.333 | 0.111 | -1.18519 | -0.03704 = - 1/3^3 |
| 0 | 0 | 0 | 0 | 0 |
| 0.2 | -0.2 | 0.04 | 0.256 | 0.008 = - 1/5^3 |
| 0 | -0.236 | 0.056 | 0 | 0 |
| -0.143 | -0.143 | 0.02 | -0.09329 | -0.00292 = - 1/7^3 |

The table shows that 1/2 times $\lambda_n$ equals the simple series $\left(1-\frac{1}{27}+\frac{1}{125}-\frac{1}{343}+\dots\right)$ or $\lambda_n = 2\left(1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\dots\right)$

## FIRST ORDER (PI RATIOS)

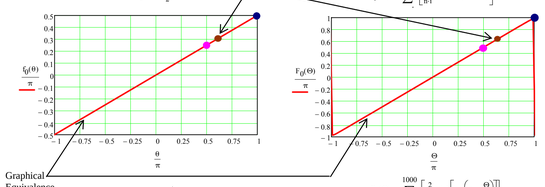
XI

**Summary of Some of the Simple Harmonics**

In this comparison it is more instructive to express the first order relation in terms of Θ/2 and θ/2 and display only the positive portion of the maps. We regraph the mapping relation from Section III in these terms with no change in result.

LHS of GTE → Real Space $f_0(\theta) := \frac{\theta}{2}$ ← Typical Point of Evaluation ← RHS of GTE → Harmonic Space $F_0(\Theta) := \sum_{n=1}^{1000}\left[\frac{2}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot(\pi-\Theta)\right]\right]$

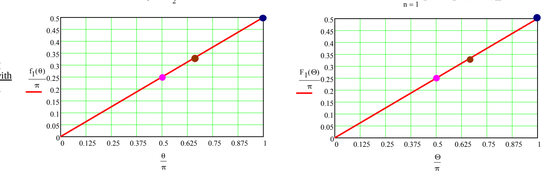
Original Map




Graphical Equivalence

$f_1(\theta) := \frac{\theta}{2}$ $F_1(\Theta) := \sum_{n=1}^{1000}\left[\frac{2}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\right]$

Regraphed for Comparison with Higher Orders




The evaluation points above get substituted into the Transform relation. The LHS is a constant in Real Space and the RHS is an infinite series. The graphical equivalence demonstrates the mapping results are equal.

$$\frac{\theta}{2} = \sum_{n=1}^{\infty}\left[\frac{2}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\right] \quad \text{(Transform)}$$

(Real) (Harmonic)

Normalization Factor results in Simple Series

Reciprocal Displays Easily Recognized Integers Ocurring in Denominator of Series

Reciprocal of Normalization Factor ocurs when solved Algebraically

Normalized Simple Series

Reciprocal of Normalized Series

$\frac{1}{2}\left[\frac{2}{n\cdot i}\cdot\sinh\left(i\cdot n\cdot\frac{\pi}{2}\right)\right] = \left[\frac{1}{2}\left[\frac{2}{n\cdot i}\cdot\sinh\left(i\cdot n\cdot\frac{\pi}{2}\right)\right]\right]^{-1} =$

| | |
|---|---|
| 1 | 1 |
| 0 | |
| -0.3333 | -3 |
| 0 | |
| 0.2 | 5 |
| 0 | ⋮ |
| -0.1429 | -7 |
| 0 | |

n = 1 ↓ n = 8

Evaluation at π

$\frac{\pi}{2} = 2x$

$\frac{\pi}{2} = 2\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots\right)$

GTE Derived Identity → $\frac{\pi}{4} = \left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots\right) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)}$

Mathematical Check → $\frac{\pi}{4} = 0.7854$ $\sum_{k=1}^{10000}\frac{(-1)^{k-1}}{(2k-1)} = 0.7854$

Normalized Simple Series

Reciprocal of Normalized Series

$\frac{1}{\sqrt{3}}\left[\frac{2}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{3}\right)\right]\right] = \left[\frac{1}{\sqrt{3}}\left[\frac{2}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{3}\right)\right]\right]\right]^{-1} =$

| | |
|---|---|
| 1 | 1 |
| -0.5 | -2 |
| 0 | |
| 0.25 | 4 |
| -0.2 | -5 |
| 0 | |
| 0.1429 | 7 |
| -0.125 | -8 |

Evaluation at 2π/3

$\frac{\pi}{3} = \sqrt{3}x$

$\frac{\pi}{3} = \sqrt{3}\left(1-\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+\dots\right) = \sqrt{3}-\left[\sqrt{3}\left(\frac{1}{3}-\frac{2}{4}+\frac{2}{5}-\frac{1}{6}+\dots\right)\right]$

$\frac{\pi}{3\cdot\sqrt{3}} = \left(1-\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+\dots\right) = \sum_{k=1}^{\infty}\left[\frac{1}{(3k-2)}-\frac{1}{(3k-1)}\right] = \sum_{k=1}^{\infty}\left[\frac{(-1)^{k-1}}{k}-\frac{(-1)^{k-1}}{3k}+\frac{2}{6k-2}-\frac{2}{6k-1}\right]$

$\frac{\pi}{3\cdot\sqrt{3}} = 0.6046$ $\sum_{k=1}^{10000}\left[\frac{1}{(3k-2)}-\frac{1}{(3k-1)}\right] = 0.6046$ $\sum_{k=1}^{10000}\left[\frac{(-1)^{k-1}}{k}-\frac{(-1)^{k-1}}{3k}+\frac{2}{6k-2}-\frac{2}{6k-1}\right] = 0.6046$

Since $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k} = \ln(2)$ (shown later) then $\ln(2) - \frac{\pi}{3\cdot\sqrt{3}} = \sum_{k=1}^{\infty}\left[\frac{(-1)^{k-1}}{3k}+\frac{2}{6k-1}-\frac{2}{6k-2}\right]$

$\ln(2)-\frac{\pi}{3\cdot\sqrt{3}} = 0.0885$ $\sum_{k=1}^{10000}\left[\frac{(-1)^{k-1}}{3k}+\frac{2}{6k-1}-\frac{2}{6k-2}\right] = 0.0885$

## SECOND ORDER (PI SQUARED RATIOS)

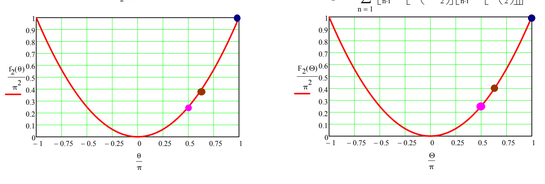
By graphing the square root of both the Real and Harmonic maps for the second order we can get a linear relationship for comparison to the first order. We regraph the mapping relation from Section X with no change in result.

Real Space

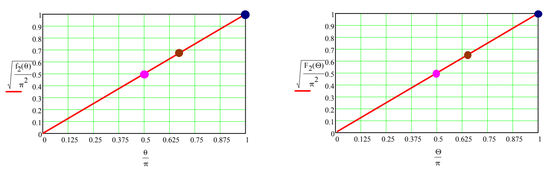
Harmonic Space

Original Map

$f_2(\theta) := \theta^2$ $F_2(\Theta) := \sum_{n=1}^{1000}\left[\frac{8}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\Theta}{2}\right)\right]\right]\right]$




Regraphed Linear for Comparison with First Order




$$\theta^2 = \sum_{n=1}^{\infty}\left[\frac{8}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\Theta}{2}\right)\right]\right]\right] \quad \text{(Transform)}$$

(Real) (Harmonic)

Normalized Simple Series

Reciprocal of Normalized Series

$\frac{1}{8}\cdot\frac{8}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right]\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{2}\right)\right]\right] = \left[\frac{1}{8}\cdot\frac{8}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right]\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{2}\right)\right]\right]\right]^{-1}$

| | |
|---|---|
| 1 | 1 |
| 0 | |
| 0.1111 | 9 |
| 0 | |
| 0.04 | 25 |
| 0 | |
| 0.0204 | 49 |
| 0 | |

Evaluation at π

$\pi^2 = 8x$

$\pi^2 = 8\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots\right)$

$\frac{\pi^2}{8} = \left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots\right) = \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$

$\frac{\pi^2}{8} = 1.2337$ $\sum_{k=1}^{10000}\frac{1}{(2k-1)^2} = 1.2337$

Normalized Simple Series

Reciprocal of Normalized Series

$\frac{4}{3}\cdot\frac{1}{8}\cdot\frac{8}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{2}{3}\pi\right)\right]\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{2}{3}\pi\right)\right]\right] = \left[\frac{4}{3}\cdot\frac{1}{8}\cdot\frac{8}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{2}{3}\pi\right)\right]\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{2}{3}\pi\right)\right]\right]\right]^{-1}$

| | |
|---|---|
| 1 | 1 |
| -0.25 | -4 |
| 0 | |
| -0.0625 | -16 |
| 0.04 | 25 |
| 0 | ⋮ |
| 0.0204 | 49 |
| -0.0156 | -64 |

Evaluation at 2π/3

$\left(\frac{2\pi}{3}\right)^2 = \frac{3}{4}x\,8\,x$

$\left(\frac{2}{3}\cdot\pi\right)^2 = \frac{3}{4}\cdot 8\left(1-\frac{1}{4}-\frac{1}{16}+\frac{1}{25}+\dots\right)$

$\frac{2}{27}\cdot\pi^2 = \left(1-\frac{1}{2^2}-\frac{1}{4^2}+\frac{1}{5^2}+\dots\right) = \sum_{k=1}^{\infty}\left[\frac{(-1)^{k-1}}{(3k-2)^2}-\frac{(-1)^{k-1}}{(3k-1)^2}\right] = \sum_{k=1}^{\infty}\left[\frac{(-1)^{k-1}}{k^2}-\frac{(-1)^{k-1}}{(3k)^2}\right]$

$\frac{2}{27}\pi^2 = 0.7311$ $\sum_{k=1}^{10000}\left[\frac{(-1)^{k-1}}{(3k-2)^2}-\frac{(-1)^{k-1}}{(3k-1)^2}\right] = 0.7311$ $\sum_{k=1}^{10000}\left[\frac{(-1)^{k-1}}{k^2}-\frac{(-1)^{k-1}}{(3k)^2}\right] = 0.7311$

Since $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2} = \frac{\pi^2}{12}$ (shown later) then $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(3k)^2} = \frac{\pi^2}{12}-\frac{2\cdot\pi^2}{27} = \frac{\pi^2}{108}$

$\frac{\pi^2}{108} = \left(\frac{1}{3^2}-\frac{1}{6^2}+\frac{1}{9^2}-\frac{1}{12^2}+\dots\right) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(3k)^2}$

$\frac{\pi^2}{108} = 0.09139$ $\sum_{k=1}^{10000}\frac{(-1)^{k-1}}{(3k)^2} = 0.09139$

## THIRD ORDER (PI CUBED RATIOS)

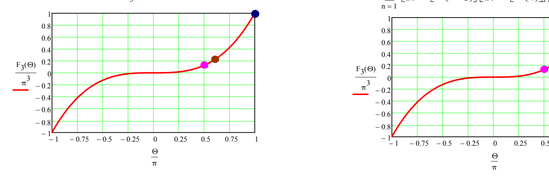
By graphing the cube root of both the Real and Harmonic maps for the second order we can get a linear relationship for comparison to the first order. We regraph the mapping relation from Section X with no change in result.

Real Space

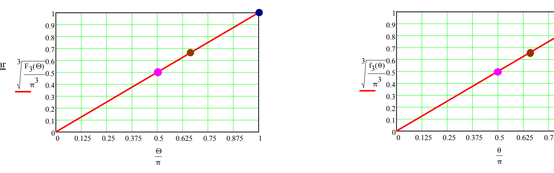
Harmonic Space

Original Map

$f_3(\theta) := \theta^3$ $F_3(\Theta) := \sum_{n=1}^{1000}\left[\frac{32}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\Theta}{4}\right)\right]\right]\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\Theta}{4}\right)\right]\right]\right]$




Regraphed Linear for Comparison




$$\theta^3 = \sum_{n=1}^{\infty}\left[\frac{32}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\Theta}{4}\right)\right]\right]\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\Theta}{4}\right)\right]\right]\right] \quad \text{(Transform)}$$

(Real) (Harmonic)

Normalized Simple Series

Reciprocal

$\left(\frac{1}{32}\cdot\frac{32}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right]\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]\right]^{-1} =$

| | |
|---|---|
| 1 | 1 |
| 0 | |
| -0.03704 | -27 |
| 0 | |
| 0.008 | 125 |
| 0 | |
| -0.00292 | -343 |
| 0 | |

Evaluation at π

$\pi^3 = 32x$

$\pi^3 = 32\left(1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\dots\right)$

$\frac{\pi^3}{32} = \left(1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\dots\right) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^3}$

$\frac{\pi^3}{32} = 0.96893$ $\sum_{k=1}^{10000}\frac{(-1)^{k-1}}{(2k-1)^3} = 0.96895$

Partially Normalized Simple Series

Reciprocal

$\frac{4}{3\cdot\sqrt{3}}\cdot\frac{1}{32}\cdot\frac{32}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{2}{3}\pi\right)\right]\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{2}{3}\pi\right)\right]\right]\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{2}{3}\pi\right)\right]\right]^2 = \left[\frac{4}{3\cdot\sqrt{3}}\cdot\frac{1}{32}\cdot\frac{32}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{2}{3}\pi\right)\right]\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{2}{3}\pi\right)\right]\right]\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{2}{3}\pi\right)\right]\right]^2\right]^{-1}$

| | |
|---|---|
| 0.33333 | 3 |
| -0.125 | -8 |
| 0 | |
| 0.01562 | 64 |
| -0.00267 | -375 |
| 0 | |
| 0.00097 | 1029 |
| -0.00195 | -512 |

Evaluation at 2π/3

$\left(\frac{2}{3}\pi\right)^3 = \frac{3\cdot\sqrt{3}}{4}x\,32\,x$

$\left(\frac{2}{3}\pi\right)^3 = 24\cdot\sqrt{3}\left(\frac{1}{3}-\frac{1}{2^3}+\frac{1}{4^3}-\frac{1}{5^3}+\dots\right)$

$\frac{8}{27}\pi^3 = 8\cdot\sqrt{3}\left(1-\frac{3}{3^3}+\frac{3}{4^3}-\frac{1}{5^3}+\dots\right)$

$\frac{1}{27\cdot\sqrt{3}}\pi^3 = \left(1-\frac{3}{3^3}+\frac{3}{5^3}-\frac{1}{7^3}+\dots\right) = \sum_{k=1}^{\infty}\left[\frac{1}{(6k-5)^3}-\frac{3}{(6k-4)^3}+\frac{3}{(6k-2)^3}-\frac{1}{(6k-1)^3}\right]$

$\frac{1}{27\cdot\sqrt{3}}\pi^3 = 0.6630$ $\sum_{k=1}^{10000}\left[\frac{1}{(6k-5)^3}-\frac{3}{(6k-4)^3}+\frac{3}{(6k-2)^3}-\frac{1}{(6k-1)^3}\right] = 0.6630$

Algebraic substitutions using $\frac{\pi^3}{32} = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^3}$ did not result in any simplifies forms.