XIII

Some First Order (Pi and ln) Relations (continued)

We can create two additional series that are more complex by evaluating the maps at 3π/4 & π/4. Since the Real and Harmonic Space maps yield the same result when evlaueted at a specific point, then the difference of two such evaluations will also yield the same result. If we atke the difference of the 3π/4 & π/4 evaluations an unwanted series of terms common to both drops out and the wanted terms double.

Evaluation at 3π/4

Real Space | Harmonic Space

$$\frac{-\ln\left(1+e^{\,i\cdot\frac{3\pi}{4}}\right)-\ln\left(1+e^{\,-i\cdot\frac{3\pi}{4}}\right)}{2} = \sum_{n=1}^{\infty}\left[\frac{1}{n}\cdot\cos\left[i\cdot n\cdot\left(\pi-\frac{3\pi}{4}\right)\right]\right]$$

$$\frac{-\ln\left(1+e^{\,i\cdot\frac{3\pi}{4}}\right)-\ln\left(1+e^{\,-i\cdot\frac{3\pi}{4}}\right)}{2} = \frac{1}{\sqrt{2}}\left(1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\right)-\frac{1}{4}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\right)$$

A normalization factor of $\sqrt{2}$ is required in order to obtain the simple series.

$\frac{-\ln\left(1+e^{\,i\cdot\frac{3\pi}{4}}\right)-\ln\left(1+e^{\,-i\cdot\frac{3\pi}{4}}\right)}{2}$ = $\frac{1}{\sqrt{2}}$ x	Normalized Simple Series $\left(\sqrt{2}\right)\left[\frac{1}{n}\cdot\cos\left[i\cdot n\cdot\left(\pi-\frac{3\pi}{4}\right)\right]\right]$ =	Reciprocal of Normalized Series $\left[\left(\sqrt{2}\right)\left[\frac{1}{n}\cdot\cos\left[i\cdot n\cdot\left(\pi-\frac{3\pi}{4}\right)\right]\right]\right]^{-1}$ =
	1	1
	0	
	-0.3333	-3
	-0.3536	-2.8284
	-0.2	-5
	0	
	0.1429	7
	0.1768	5.6569

$$\frac{-\ln\left(1+e^{\,i\cdot\frac{3\pi}{4}}\right)-\ln\left(1+e^{\,-i\cdot\frac{3\pi}{4}}\right)}{2} = \frac{1}{\left(\sqrt{2}\right)}\cdot\left[\left(1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\right)-\frac{1}{2\sqrt{2}}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\right)\right]$$

$$\left(\frac{\sqrt{2}}{2}\right)\left(-\ln\left(1+e^{\,i\cdot\frac{3\pi}{4}}\right)-\ln\left(1+e^{\,-i\cdot\frac{3\pi}{4}}\right)\right) = \left(1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\right)-\frac{1}{2\sqrt{2}}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n-3}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n-1}-\frac{\sqrt{2}}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

$\left(\frac{\sqrt{2}}{2}\right)\left(-\ln\left(1+e^{\,i\cdot\frac{3\pi}{4}}\right)-\ln\left(1+e^{\,-i\cdot\frac{3\pi}{4}}\right)\right) = 0.3782$

$\left[1\cdot\sum_{n=1}^{10000}\frac{(-1)^{n-1}}{4n-3}\right]-\left[1\cdot\sum_{n=1}^{10000}\frac{(-1)^{n-1}}{4n-1}\right]-\left[\frac{1}{2\sqrt{2}}\cdot\sum_{n=1}^{10000}\frac{(-1)^{n-1}}{n}\right] = 0.3782$

Lower Map (Red):

Evaluation at π/4

Real Space | Harmonic Space

$$\frac{-\ln\left(1+e^{\,i\cdot\frac{\pi}{4}}\right)-\ln\left(1+e^{\,-i\cdot\frac{\pi}{4}}\right)}{2} = \sum_{n=1}^{\infty}\left[\frac{1}{n}\cdot\cos\left[i\cdot n\cdot\left(\pi-\frac{\pi}{4}\right)\right]\right]$$

$$\frac{-\ln\left(1+e^{\,i\cdot\frac{\pi}{4}}\right)-\ln\left(1+e^{\,-i\cdot\frac{\pi}{4}}\right)}{2} = \frac{-1}{\sqrt{2}}\left(1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\right)-\frac{1}{4}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\right)$$

A normalization factor of $-\sqrt{2}$ is required in order to obtain the simple series.

$\frac{-\ln\left(1+e^{\,i\cdot\frac{\pi}{4}}\right)-\ln\left(1+e^{\,-i\cdot\frac{\pi}{4}}\right)}{2}$ = $\frac{1}{(-\sqrt{2})}$ x	Normalized Simple Series $(-\sqrt{2})\left[\frac{1}{n}\cdot\cos\left[i\cdot n\cdot\left(\pi-\frac{\pi}{4}\right)\right]\right]$ =	Reciprocal of Normalized Series $\left[(-\sqrt{2})\left[\frac{1}{n}\cdot\cos\left[i\cdot n\cdot\left(\pi-\frac{\pi}{4}\right)\right]\right]\right]^{-1}$ =
	1	1
	0	
	-0.3333	-3
	0.3536	2.8284
	-0.2	-5
	0	
	0.1429	7
	-0.1768	-5.6569

$$\frac{-\ln\left(1+e^{\,i\cdot\frac{\pi}{4}}\right)-\ln\left(1+e^{\,-i\cdot\frac{\pi}{4}}\right)}{2} = \frac{1}{(-\sqrt{2})}\cdot\left[\left(1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\right)+\frac{1}{2\sqrt{2}}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\right)\right]$$

$$\left(\frac{\sqrt{2}}{2}\right)\left(\ln\left(1+e^{\,i\cdot\frac{\pi}{4}}\right)+\ln\left(1+e^{\,-i\cdot\frac{\pi}{4}}\right)\right) = \left(1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\right)+\frac{1}{2\sqrt{2}}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n-3}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n-1}+\frac{\sqrt{2}}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

$\left(\frac{\sqrt{2}}{4}\right)\left(\ln\left(1+e^{\,i\cdot\frac{\pi}{4}}\right)+\ln\left(1+e^{\,-i\cdot\frac{\pi}{4}}\right)\right) = 0.4341$

$\left[\sum_{n=1}^{10000}\frac{(-1)^{n-1}}{4n-3}\right]-\left[\sum_{n=1}^{10000}\frac{(-1)^{n-1}}{4n-1}\right]+\left[\frac{1}{2\sqrt{2}}\cdot\sum_{n=1}^{10000}\frac{(-1)^{n-1}}{n}\right] = 0.8683$

If we let $A = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n-3}$ $B = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n-1}$ the difference $K_2 = A - B$ is the series we just determined:

$$K_2 = A - B = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n-3}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n-1} = \frac{\sqrt{2}}{4}\left[\ln\left(1-e^{\,i\cdot\frac{3\pi}{4}}\right)+\ln\left(1-e^{\,-i\cdot\frac{3\pi}{4}}\right)-\left(\ln\left(1-e^{\,i\cdot\frac{\pi}{4}}\right)+\ln\left(1-e^{\,-i\cdot\frac{\pi}{4}}\right)\right)\right]$$

And $K_1 = A + B = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n-3}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n-1} = \frac{\pi}{2\sqrt{2}}$ (already Shown on Sheet 3.1)

So $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n-3} = \frac{K_1+K_2}{2}$ and $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n-1} = \frac{K_1-K_2}{2}$

Evaluation at 3π/4 - π/4

Real Space | Harmonic Space

$$\frac{-\ln\left(1+e^{\,i\cdot\frac{3\pi}{4}}\right)-\ln\left(1+e^{\,-i\cdot\frac{3\pi}{4}}\right)}{2}-\frac{-\ln\left(1+e^{\,i\cdot\frac{\pi}{4}}\right)-\ln\left(1+e^{\,-i\cdot\frac{\pi}{4}}\right)}{2} = \sum_{n=1}^{\infty}\left[\frac{1}{n}\cdot\cos\left[i\cdot n\cdot\left(\pi-\frac{3\pi}{4}\right)\right]\right]-\sum_{n=1}^{\infty}\left[\frac{1}{n}\cdot\cos\left[i\cdot n\cdot\left(\pi-\frac{\pi}{4}\right)\right]\right]$$

$$\frac{-\ln\left(1+e^{\,i\cdot\frac{3\pi}{4}}\right)-\ln\left(1+e^{\,-i\cdot\frac{3\pi}{4}}\right)+\ln\left(1+e^{\,i\cdot\frac{\pi}{4}}\right)+\ln\left(1+e^{\,-i\cdot\frac{\pi}{4}}\right)}{2} = \frac{1}{\sqrt{2}}\left(1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\right)-\frac{1}{4}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\right) \quad \frac{-1}{\sqrt{2}}\left(1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\right)-\frac{1}{4}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\right) = \frac{2}{\sqrt{2}}\left(1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\right)$$

A normalization factor of $\frac{\sqrt{2}}{2}$ is required in order to obtain the simple series.

$\frac{-\ln\left(1+e^{\,i\cdot\frac{3\pi}{4}}\right)-\ln\left(1+e^{\,-i\cdot\frac{3\pi}{4}}\right)+\ln\left(1+e^{\,i\cdot\frac{\pi}{4}}\right)+\ln\left(1+e^{\,-i\cdot\frac{\pi}{4}}\right)}{2}$ = $\left(\frac{2}{\sqrt{2}}\right)$ x	Normalized Simple Series $\left(\frac{\sqrt{2}}{2}\right)\left[\frac{1}{n}\cdot\cos\left[i\cdot n\cdot\left(\pi-\frac{3\pi}{4}\right)\right]-\frac{1}{n}\cdot\cos\left[i\cdot n\cdot\left(\pi-\frac{\pi}{4}\right)\right]\right]$ =	Reciprocal of Normalized Series $\left[\left(2\sqrt{2}\right)\left[\frac{1}{n}\cdot\cos\left[\left(\frac{\sqrt{2}}{2}\right)\left[\frac{1}{n}\cdot\cos\left[i\cdot n\cdot\left(\pi-\frac{3\pi}{4}\right)\right]-\frac{1}{n}\cdot\cos\left[i\cdot n\cdot\left(\pi-\frac{\pi}{4}\right)\right]\right]\right]\right]^{-1}$
	1	1
	0	
	-0.3333	-3
	0	
	-0.2	-5
	0	
	0.1429	7
	0	0

$$\frac{-\ln\left(1+e^{\,i\cdot\frac{3\pi}{4}}\right)-\ln\left(1+e^{\,-i\cdot\frac{3\pi}{4}}\right)+\ln\left(1+e^{\,i\cdot\frac{\pi}{4}}\right)+\ln\left(1+e^{\,-i\cdot\frac{\pi}{4}}\right)}{2} = \left(\frac{2}{\sqrt{2}}\right)\cdot\left(1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+-\right)$$

$$\left(\frac{\sqrt{2}}{2}\right)\left(-\ln\left(1+e^{\,i\cdot\frac{3\pi}{4}}\right)-\ln\left(1+e^{\,-i\cdot\frac{3\pi}{4}}\right)+\ln\left(1+e^{\,i\cdot\frac{\pi}{4}}\right)+\ln\left(1+e^{\,-i\cdot\frac{\pi}{4}}\right)\right) = \left(1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+-\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n-3}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n-1}$$

$\left(\frac{\sqrt{2}}{4}\right)\left[\ln\left(1-e^{\,i\cdot\frac{3\pi}{4}}\right)+\ln\left(1-e^{\,-i\cdot\frac{3\pi}{4}}\right)-\left(\ln\left(1-e^{\,i\cdot\frac{\pi}{4}}\right)+\ln\left(1-e^{\,-i\cdot\frac{\pi}{4}}\right)\right)\right] = 0.6232$

$\sum_{n=1}^{1000}\left[\frac{(-1)^{n-1}}{4n-3}-\frac{(-1)^{n-1}}{4n-1}\right] = 0.6232$

It is possible to separate this series into the difference of two alternating series and then algebraically seperate them.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n-3} = \left(1-\frac{1}{5}+\frac{1}{9}-\frac{1}{13}-\right) = \frac{1}{4\sqrt{2}}\left[\pi+\left[\ln\left(1-e^{\,i\cdot\frac{3\pi}{4}}\right)+\ln\left(1-e^{\,-i\cdot\frac{3\pi}{4}}\right)-\left(\ln\left(1-e^{\,i\cdot\frac{\pi}{4}}\right)+\ln\left(1-e^{\,-i\cdot\frac{\pi}{4}}\right)\right)\right]\right]$$

$\sum_{n=1}^{100000}\frac{(-1)^{n-1}}{4n-3} = 0.867$ $\frac{1}{4\sqrt{2}}\left[\pi+\left[\ln\left(1-e^{\,i\cdot\frac{3\pi}{4}}\right)+\ln\left(1-e^{\,-i\cdot\frac{3\pi}{4}}\right)-\left(\ln\left(1-e^{\,i\cdot\frac{\pi}{4}}\right)+\ln\left(1-e^{\,-i\cdot\frac{\pi}{4}}\right)\right)\right]\right] = 0.867$

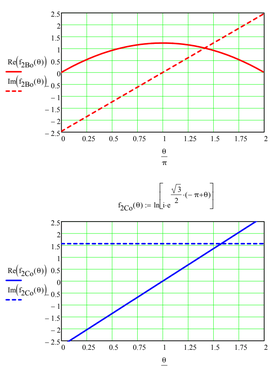
$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n-1} = \left(\frac{1}{3}-\frac{1}{7}+\frac{1}{11}-\frac{1}{15}-\right) = \frac{1}{4\sqrt{2}}\left[\pi-\left[\ln\left(1-e^{\,i\cdot\frac{3\pi}{4}}\right)+\ln\left(1-e^{\,-i\cdot\frac{3\pi}{4}}\right)-\left(\ln\left(1-e^{\,i\cdot\frac{\pi}{4}}\right)+\ln\left(1-e^{\,-i\cdot\frac{\pi}{4}}\right)\right)\right]\right]$$

$\sum_{n=1}^{100000}\frac{(-1)^{n-1}}{4n-1} = 0.2437$ $\frac{1}{4\sqrt{2}}\left[\pi-\left[\ln\left(1-e^{\,i\cdot\frac{3\pi}{4}}\right)+\ln\left(1-e^{\,-i\cdot\frac{3\pi}{4}}\right)-\left(\ln\left(1-e^{\,i\cdot\frac{\pi}{4}}\right)+\ln\left(1-e^{\,-i\cdot\frac{\pi}{4}}\right)\right)\right]\right] = 0.2437$

We examined this similar series as well but it did not provide any new identities.

Original Function

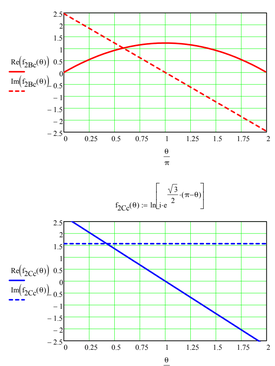
$f_{2Bo}(\theta) := \frac{-1}{2}\cdot\ln\left[1\cdot e^{\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}\right]^2$



$f_{2Co}(\theta) := \ln\left[1\cdot e^{\frac{\sqrt{3}}{2}\cdot(-\pi+\theta)}\right]$

Conjugate

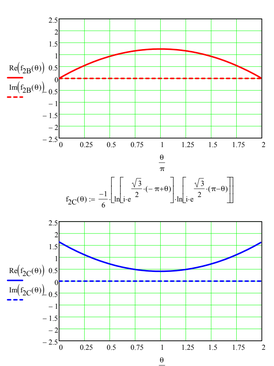
$f_{2Bc}(\theta) := \frac{-1}{2}\cdot\left[\ln\left[1\cdot e^{\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}\right]\right]^2$



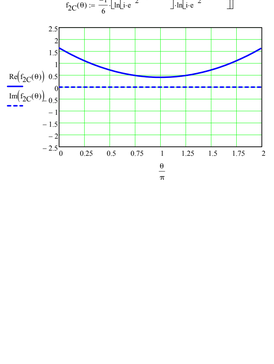
$f_{2Cc}(\theta) := \ln\left[1\cdot e^{\frac{\sqrt{3}}{2}\cdot(\pi-\theta)}\right]$

Final Real Space Mapping Function

$f_{2B}(\theta) := \frac{-1}{4}\cdot\left[\left[\ln\left[1\cdot e^{\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}\right]\right]^2+\left[\ln\left[1\cdot e^{-\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}\right]\right]^2\right]$



$f_{2C}(\theta) := \frac{-1}{6}\cdot\left[\left[\ln\left[1\cdot e^{\frac{\sqrt{3}}{2}\cdot(-\pi+\theta)}\right]\right]\cdot\left[\ln\left[1\cdot e^{\frac{\sqrt{3}}{2}\cdot(\pi-\theta)}\right]\right]\right]$



Evaluating the upper (red) function at 0, π/2, and π:

$\frac{-1}{2}\cdot\ln\left[1\cdot e^{\left(\frac{\pi}{2}-\frac{0}{2}\right)}\right]^2 = -i\cdot\frac{\pi^2}{4}$

$\frac{-1}{2}\cdot\left[\ln\left[1\cdot e^{-\left(\frac{\pi}{2}-\frac{0}{2}\right)}\right]\right]^2 = i\cdot\frac{\pi^2}{4}$

$\frac{-1}{4}\cdot\left[\left[\ln\left[1\cdot e^{\left(\frac{\pi}{2}-\frac{0}{2}\right)}\right]\right]^2+\left[\ln\left[1\cdot e^{-\left(\frac{\pi}{2}-\frac{0}{2}\right)}\right]\right]^2\right] = 0$

$\frac{-1}{2}\cdot\left[\ln\left[1\cdot e^{\left(\frac{\pi}{2}-\frac{\pi}{2}\right)}\right]\right]^2 = \frac{3}{32}\cdot\pi^2-i\cdot\frac{\pi^2}{8}$

$\frac{-1}{2}\cdot\left[\left[\ln\left[1\cdot e^{-\left(\frac{\pi}{2}-\frac{\pi}{2}\right)}\right]\right]\right]^2 = \frac{3}{32}\cdot\pi^2+i\cdot\frac{\pi^2}{8}$

$\frac{-1}{4}\cdot\left[\left[\ln\left[1\cdot e^{\left(\frac{\pi}{2}-\frac{\pi}{2}\right)}\right]\right]^2+\left[\left[\ln\left[1\cdot e^{-\left(\frac{\pi}{2}-\frac{\pi}{2}\right)}\right]\right]\right]^2\right] = \frac{3}{32}\cdot\pi^2$

$\frac{-1}{2}\cdot\left[\ln\left[1\cdot e^{\left(\frac{\pi}{2}-\frac{\pi}{2}\right)}\right]\right]^2 = \frac{\pi^2}{8}$

$\frac{-1}{2}\cdot\left[\left[\ln\left[1\cdot e^{-\left(\frac{\pi}{2}-\frac{\pi}{2}\right)}\right]\right]\right]^2 = \frac{\pi^2}{8}$

$\frac{-1}{4}\cdot\left[\left[\ln\left[1\cdot e^{\left(\frac{\pi}{2}-\frac{\pi}{2}\right)}\right]\right]^2+\left[\left[\ln\left[1\cdot e^{-\left(\frac{\pi}{2}-\frac{\pi}{2}\right)}\right]\right]\right]^2\right] = \frac{\pi^2}{8}$

Evaluating the lower (blue) function at 0, π/2, and π:

$\left(\ln\left(1+e^{\,i\cdot 0}\right)\right)_{e^{-i\cdot 0}} = \ln(2)$

$\left(\ln\left(1+e^{-i\cdot 0}\right)\right)_{e^{i\cdot 0}} = \ln(2)$

$\frac{\left(\ln\left(1+e^{\,i\cdot 0}\right)\right)_{e^{-i\cdot 0}}+\left(\ln\left(1+e^{-i\cdot 0}\right)\right)_{e^{i\cdot 0}}}{2} = \ln(2)$

$\left(\ln\left(1+e^{\,i\cdot\frac{\pi}{2}}\right)\right)_{e^{-i\cdot\frac{\pi}{2}}} = \pi/4 - i\ln(2)/2$

$\left(\ln\left(1+e^{-i\cdot\frac{\pi}{2}}\right)\right)_{e^{i\cdot\frac{\pi}{2}}} = \pi/4 + i\ln(2)/2$

$\frac{\left(\ln\left(1+e^{\,i\cdot\frac{\pi}{2}}\right)\right)_{e^{-i\cdot\frac{\pi}{2}}}+\left(\ln\left(1+e^{-i\cdot\frac{\pi}{2}}\right)\right)_{e^{i\cdot\frac{\pi}{2}}}}{2} = \pi/4$

$\left(\ln\left(1+e^{\,i\cdot\pi}\right)\right)_{e^{-i\cdot\pi}} = \infty - i\,\pi/2$

$\left(\ln\left(1+e^{-i\cdot\pi}\right)\right)_{e^{i\cdot\pi}} = \infty + i\,\pi/2$

$\frac{\left(\ln\left(1+e^{\,i\cdot\pi}\right)\right)_{e^{-i\cdot\pi}}+\left(\ln\left(1+e^{-i\cdot\pi}\right)\right)_{e^{i\cdot\pi}}}{2} = \infty$