Casey's Curves

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XII

Introduction to Complex Logarithmic Harmonics

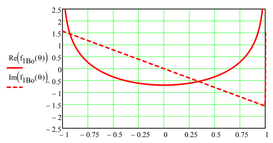
This section involves functions utilizing the natural logarithm of imaginary numbers. These produce functions that have a real and imaginary components. Each function has a conjugate. By adding or subtracting the conjugate from the original function the imaginary component is negated and drops out and the real component doubles. It is then divided by 2 for normalization. Since the function yields real values only, it is observeable. This observeable function is part of the solution set for the transform equation.

The resultant functions are used as the Real Space maps in the transforms (next). This section shows how the two Real Space maps are developed. The colored dots on the map represent specific the evaluation points that are evaluated at the lower right.

Development of Real Space Maps

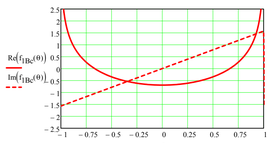
Original Function

$f_{IBo}(\theta) := \ln\left(1 + e^{i\cdot\theta}\right)$



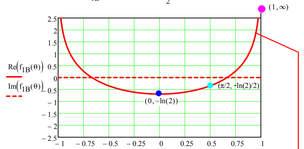
Conjugate

$f_{IBc}(\theta) := -\ln\left(1 + e^{-i\cdot\theta}\right)$

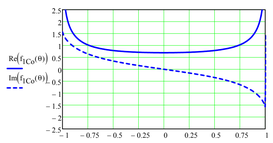


Final Real Space Mapping Function

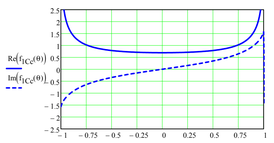
$f_{IB}(\theta) := \frac{-\ln\left(1 + e^{i\cdot\theta}\right) - \ln\left(1 + e^{-i\cdot\theta}\right)}{2}$



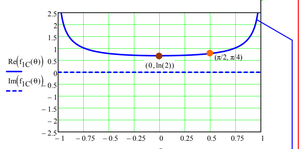
$f_{ICo}(\theta) := \left(\ln\left(1 + e^{i\cdot\theta}\right)\right)\cdot e^{-i\cdot\theta}$



$f_{ICc}(\theta) := \left(\ln\left(1 + e^{-i\cdot\theta}\right)\right)\cdot e^{i\cdot\theta}$



$f_{IC}(\theta) := \frac{\left(\ln\left(1 + e^{i\cdot\theta}\right)\right)\cdot e^{-i\cdot\theta} + \left(\ln\left(1 + e^{-i\cdot\theta}\right)\right)\cdot e^{i\cdot\theta}}{2}$



Logarithms of Imaginary & Complex Numbers

Inverse Euler Functions

From the Euler unit circle we have inverse compass readings for +1, +i, -1, -i that result in 0, $\frac{\pi}{2}\cdot i$, $\pi\cdot i$, and $-\frac{\pi}{2}\cdot i$ as shown on Sheet 2.

Logarithmic Inverse	Euler Function
$\ln(1) = 0$	$e^{i\cdot 0} = 1$
$\ln(i) = \frac{\pi}{2}\cdot i$	$e^{\frac{\pi}{2}\cdot i} = i$
$\ln(-1) = \pi\cdot i$	$e^{\pi\cdot i} = -1$
$\ln(-i) = -\frac{\pi}{2}\cdot i$	$e^{-\frac{\pi}{2}\cdot i} = -i$

Natural Log of a Negative Number

Use $e^{s+t} = e^{s}\cdot e^{t}$, then assume $t = \pi\cdot i$

then $e^{\ln(a)+\pi\cdot i} = e^{\ln(a)}\cdot e^{\pi\cdot i} = (a)\cdot(-1) = -a$

So $\ln(-a) = \ln\left(e^{\ln(a)+\pi\cdot i}\right) = \ln(a) + \pi\cdot i$

Logarithm	Euler Inverse
$\ln(-1) = \pi\cdot i$	$e^{\pi\cdot i} = -1$
$\ln(-2) = \ln(2) + \pi\cdot i$	$e^{\ln(2)+\pi\cdot i} = -2$
$\ln(-a) = \ln(a) + \pi\cdot i$	$e^{a+\pi\cdot i} = -e^{a}$

Natural Log of a Positive Imaginary Number

Use $e^{s+t} = e^{s}\cdot e^{t}$, then assume $t = \frac{\pi}{2}\cdot i$

then $e^{\ln(a)+\frac{\pi}{2}\cdot i} = e^{\ln(a)}\cdot e^{\frac{\pi}{2}\cdot i} = (a)\cdot(i) = a\cdot i$

So $\ln(a\cdot i) = \ln\left(e^{\ln(a)+\frac{\pi}{2}\cdot i}\right) = \ln(a) + \frac{\pi}{2}\cdot i$

Logarithm	Euler Inverse
$\ln(i) = \frac{\pi}{2}\cdot i$	$e^{\frac{\pi}{2}\cdot i} = i$
$\ln(2\cdot i) = \ln(2) + \frac{\pi}{2}\cdot i$	$e^{\ln(2)+\frac{\pi}{2}\cdot i} = 2i$
$\ln(a\cdot i) = \ln(a) + \frac{\pi}{2}\cdot i$	$e^{\ln(a)+\frac{\pi}{2}\cdot i} = a\cdot i$

Natural Log of a Negative Imaginary Number

Use $e^{s+t} = e^{s}\cdot e^{t}$, then assume $t = -\frac{\pi}{2}\cdot i$

then $e^{\ln(a)-\frac{\pi}{2}\cdot i} = e^{\ln(a)}\cdot e^{-\frac{\pi}{2}\cdot i} = (a)\cdot(-i) = -a\cdot i$

So $\ln(-a\cdot i) = \ln\left(e^{\ln(a)-\frac{\pi}{2}\cdot i}\right) = \ln(a) - \frac{\pi}{2}\cdot i$

Logarithm	Euler Inverse
$\ln(-i) = -\frac{\pi}{2}\cdot i$	$e^{-\frac{\pi}{2}\cdot i} = -i$
$\ln(-2\cdot i) = \ln(2) - \frac{\pi}{2}\cdot i$	$e^{\ln(2)-\frac{\pi}{2}\cdot i} = -2i$
$\ln(-a\cdot i) = \ln(a) - \frac{\pi}{2}\cdot i$	$e^{\ln(a)-\frac{\pi}{2}\cdot i} = -a\cdot i$

Natural Log of a Complex Number

Use $e^{s+t} = e^{s}\cdot e^{t}$, with $s = \ln\left(\sqrt{a^2+b^2}\right)$ and $t = \operatorname{atan}\left(\frac{b}{a}\right)$

So $e^{\ln\left(\sqrt{a^2+b^2}\right)+\operatorname{atan}\left(\frac{b}{a}\right)\cdot i} = e^{\ln\left(\sqrt{a^2+b^2}\right)}\cdot e^{i\cdot\operatorname{atan}\left(\frac{b}{a}\right)}$

$\operatorname{atan}\left(\frac{b}{a}\right)$ is an angle on the unit circle and $e^{i\cdot\operatorname{atan}\left(\frac{b}{a}\right)}$ and are its real & imaginary components $\frac{a}{\sqrt{a^2+b^2}}$ and $i\cdot\frac{b}{\sqrt{a^2+b^2}}$

So $e^{\ln\left(\sqrt{a^2+b^2}\right)+\operatorname{atan}\left(\frac{b}{a}\right)\cdot i} = e^{\ln\left(\sqrt{a^2+b^2}\right)}\left(\frac{a}{\sqrt{a^2+b^2}} + i\cdot\frac{b}{\sqrt{a^2+b^2}}\right) = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}} + i\cdot\frac{b}{\sqrt{a^2+b^2}}\right) = a + b\cdot i$

Logarithm	Euler Inverse
$\ln(1 + 2\cdot i) = 0.805 + 1.107i$	$e^{.805+0.464\cdot i} = 2 + i$
$\ln(2 + 3\cdot i) = 1.282 + 0.983i$	$e^{1.282+0.983\cdot i} = 2 + 3i$
$\ln(\sqrt{5}) = 0.805$	$\operatorname{atan}\left(\frac{1}{2}\right) = 0.464$
$\ln(\sqrt{13}) = 1.282$	$\operatorname{atan}\left(\frac{2}{1}\right) = 1.107$
$e^{\ln(\sqrt{5})+i\cdot\operatorname{atan}\left(\frac{1}{2}\right)} = 2 + i$	
$e^{\ln(\sqrt{5})}\left(\frac{1}{\sqrt{5}} + i\cdot\frac{2}{\sqrt{5}}\right) = 1 + 2i$	
$e^{\ln\left(\sqrt{a^2+b^2}\right)+\operatorname{atan}\left(\frac{b}{a}\right)\cdot i} = \ln(a + b\cdot i)$	

Evaluating the upper (red) function at 0, π/2, and π:

$-\ln\left(1 + e^{i\cdot 0}\right) = -\ln(2)$

$-\ln\left(1 + e^{i\cdot\frac{\pi}{2}}\right) = -\ln(2)/2 - i\,\pi/4$

$-\ln\left(1 + e^{i\cdot\pi}\right) = \infty - i\,\pi/2$

$-\ln\left(1 + e^{-i\cdot 0}\right) = -\ln(2)$

$-\ln\left(1 + e^{-i\cdot\frac{\pi}{2}}\right) = -\ln(2)/2 + i\,\pi/4$

$-\ln\left(1 + e^{-i\cdot\pi}\right) = \infty + i\,\pi/2$

$\frac{-\ln\left(1 + e^{i\cdot 0}\right) - \ln\left(1 + e^{-i\cdot 0}\right)}{2} = -\ln(2)$

$\frac{-\ln\left(1 + e^{i\cdot\frac{\pi}{2}}\right) - \ln\left(1 + e^{-i\cdot\frac{\pi}{2}}\right)}{2} = -\ln(2)/2$

$\frac{-\ln\left(1 + e^{i\cdot\pi}\right) - \ln\left(1 + e^{-i\cdot\pi}\right)}{2} = \infty$

Evaluating the lower (blue) function at 0, π/2, and π:

$\left(\ln\left(1 + e^{i\cdot 0}\right)\right)\cdot e^{-i\cdot 0} = \ln(2)$

$\left(\ln\left(1 + e^{i\cdot\frac{\pi}{2}}\right)\right)\cdot e^{-i\cdot\frac{\pi}{2}} = \pi/4 - i\,\ln(2)/2$

$\left(\ln\left(1 + e^{i\cdot\pi}\right)\right)\cdot e^{-i\cdot\pi} = \infty - i\,\pi/2$

$\left(\ln\left(1 + e^{-i\cdot 0}\right)\right)\cdot e^{i\cdot 0} = \ln(2)$

$\left(\ln\left(1 + e^{-i\cdot\frac{\pi}{2}}\right)\right)\cdot e^{i\cdot\frac{\pi}{2}} = \pi/4 + i\,\ln(2)/2$

$\left(\ln\left(1 + e^{-i\cdot\pi}\right)\right)\cdot e^{i\cdot\pi} = \infty + i\,\pi/2$

$\frac{\left(\ln\left(1 + e^{i\cdot 0}\right)\right)\cdot e^{-i\cdot 0} + \left(\ln\left(1 + e^{-i\cdot 0}\right)\right)\cdot e^{i\cdot 0}}{2} = \ln(2)$

$\frac{\left(\ln\left(1 + e^{i\cdot\frac{\pi}{2}}\right)\right)\cdot e^{-i\cdot\frac{\pi}{2}} + \left(\ln\left(1 + e^{-i\cdot\frac{\pi}{2}}\right)\right)\cdot e^{i\cdot\frac{\pi}{2}}}{2} = \pi/4$

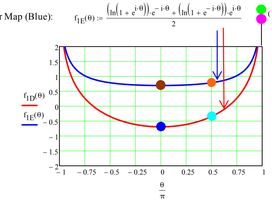
$\frac{\left(\ln\left(1 + e^{i\cdot\pi}\right)\right)\cdot e^{-i\cdot\pi} + \left(\ln\left(1 + e^{-i\cdot\pi}\right)\right)\cdot e^{i\cdot\pi}}{2} = \infty$

The Real Space maps were created from the complex functions above which were combined with the conjugate to yield a real result and then halved for normalization.

(Real)

Lower Map (Red): $f_{1D}(\theta) := \frac{-\ln\left(1 + e^{i\cdot\theta}\right) - \ln\left(1 + e^{-i\cdot\theta}\right)}{2}$

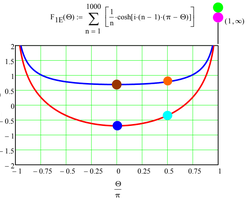
Upper Map (Blue): $f_{1E}(\theta) := \frac{\left(\ln\left(1 + e^{i\cdot\theta}\right)\right)\cdot e^{-i\cdot\theta} + \left(\ln\left(1 + e^{-i\cdot\theta}\right)\right)\cdot e^{i\cdot\theta}}{2}$



(Harmonic)

$F_{1D}(\Theta) := \sum_{n=1}^{1000}\left[\frac{1}{n}\cdot\cos\left[i\cdot n\cdot(\pi - \Theta)\right]\right]$

$F_{1E}(\Theta) := \sum_{n=1}^{1000}\left[\frac{1}{n}\cdot\cos\left[i\cdot(n-1)\cdot(\pi - \Theta)\right]\right]$



The development of these Real Space maps is shown above.

(Real) (Harmonic)

Lower Map (Red): $\frac{-\ln\left(1 + e^{i\cdot\theta}\right) - \ln\left(1 + e^{-i\cdot\theta}\right)}{2} = \sum_{n=1}^{\infty}\left[\frac{1}{n}\cdot\cos\left[i\cdot n\cdot(\pi - \Theta)\right]\right]$ (Transform)

Upper Map (Blue): $\frac{\left(\ln\left(1 + e^{i\cdot\theta}\right)\right)\cdot e^{-i\cdot\theta} + \left(\ln\left(1 + e^{-i\cdot\theta}\right)\right)\cdot e^{i\cdot\theta}}{2} = \sum_{n=1}^{\infty}\left[\frac{1}{n}\cdot\cos\left[i\cdot(n-1)\cdot(\pi - \Theta)\right]\right]$ (Transform)

XIII

Some First Order (Pi and ln) Relations

Lower Map (Red):

Evaluation at 0

Real Space	Harmonic Space
$\frac{-\ln\left(1 + e^{i\cdot 0}\right) - \ln\left(1 + e^{-i\cdot 0}\right)}{2}$	$\sum_{n=1}^{\infty}\left[\frac{1}{n}\cdot\cos\left[i\cdot n\cdot(\pi - 0)\right]\right]$
$-\ln(2)$	$\left(-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \ldots\right)$

A normalization factor of -1 is applied to both sides in order to obtain the simple series.

Normalized Simple Series	Reciprocal of Normalized Series
$\frac{1}{n}\cdot\cos\left[i\cdot n\cdot(\pi - 0)\right] =$	$\left[\frac{1}{n}\cdot\cos\left[i\cdot n\cdot(\pi - 0)\right]\right]^{-1} =$
1	1
-0.5	-2
0.3333	3
-0.25	-4
0.2	5
-0.1667	-6
0.1429	7
-0.125	-8

$-\ln(2) = \frac{1}{(-1)}\cdot x$

$-\ln(2) = \frac{1}{(-1)}\cdot x\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right)$

$\ln(2) = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}$

Evaluation at π/2

Real Space	Harmonic Space
$\frac{-\ln\left(1 + e^{i\cdot\frac{\pi}{2}}\right) - \ln\left(1 + e^{-i\cdot\frac{\pi}{2}}\right)}{2}$	$\sum_{n=1}^{\infty}\left[\frac{1}{n}\cdot\cos\left[i\cdot n\cdot(\pi - 0)\right]\right]$
$-\ln(2)/2$	$\left(-\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \frac{1}{8} + \ldots\right)$

A normalization factor of -2 is applied to both sides in order to obtain the simple series.

Normalized Simple Series	Reciprocal of Normalized Series
$(-2)\cdot\frac{1}{n}\cdot\cos\left[i\cdot n\cdot\left(\pi - \frac{\pi}{2}\right)\right] =$	$\left[(-2)\cdot\frac{1}{n}\cdot\cos\left[i\cdot n\cdot\left(\pi - \frac{\pi}{2}\right)\right]\right]^{-1} =$
0	
1	1
0	
-0.5	-2
0	
0.3333	3
0	
-0.25	-4

$-\ln(2)/2 = \frac{1}{(-2)}\cdot x$

$-\ln(2)/2 = \frac{1}{(-2)}\cdot x\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right)$

$\ln(2) = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}$

Evaluation at π

Real Space	Harmonic Space
$\frac{-\ln\left(1 + e^{i\cdot\pi}\right) - \ln\left(1 + e^{-i\cdot\pi}\right)}{2}$	$\sum_{n=1}^{\infty}\left[\frac{1}{n}\cdot\cos\left[i\cdot n\cdot(\pi - \pi)\right]\right]$
∞	$\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right)$

A normalization factor is not required in order to obtain the simple series.

Normalized Simple Series	Reciprocal of Normalized Series
$\frac{1}{n}\cdot\cos\left[i\cdot n\cdot(\pi - \pi)\right] =$	$\left[\frac{1}{n}\cdot\cos\left[i\cdot n\cdot(\pi - \pi)\right]\right]^{-1} =$
1	1
0.5	2
0.3333	3
0.25	4
0.2	5
0.1667	6
0.1429	7
0.125	8

$\infty = \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right) = \sum_{k=1}^{\infty}\frac{1}{k}$

(Does not Converge)

Upper Map (Blue):

Evaluation at 0

Real Space	Harmonic Space
$\frac{\left(\ln\left(1 + e^{i\cdot 0}\right)\right)\cdot e^{-i\cdot 0} + \left(\ln\left(1 + e^{-i\cdot 0}\right)\right)\cdot e^{i\cdot 0}}{2}$	$\sum_{n=1}^{\infty}\left[\frac{1}{n}\cdot\cos\left[i\cdot(n-1)\cdot(\pi - 0)\right]\right]$
$\ln(2)$	$\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right)$

A normalization factor is not required in order to obtain the simple series.

Normalized Simple Series	Reciprocal of Normalized Series
$\frac{1}{n}\cdot\cos\left[i\cdot(n-1)\cdot(\pi - 0)\right] =$	$\left[\frac{1}{n}\cdot\cos\left[i\cdot(n-1)\cdot(\pi - 0)\right]\right]^{-1} =$
1	1
-0.5	-2
0.3333	3
-0.25	-4
0.2	5
-0.1667	-6
0.1429	7
-0.125	-8

$\ln(2) = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}$

(No new information)

Evaluation at π/2

Real Space	Harmonic Space
$\frac{\left(\left(\ln\left(1 + e^{i\cdot\frac{\pi}{2}}\right)\right)\cdot e^{-i\cdot\frac{\pi}{2}} + \left(\left(\ln\left(1 + e^{-i\cdot\frac{\pi}{2}}\right)\right)\cdot e^{i\cdot\frac{\pi}{2}}\right.}{2}$	$\sum_{n=1}^{\infty}\left[\frac{1}{n}\cdot\cos\left[i\cdot(n-1)\cdot\left(\pi - \frac{\pi}{2}\right)\right]\right]$
$\pi/4$	$\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right)$

A normalization factor is not required in order to obtain the simple series.

Normalized Simple Series	Reciprocal of Normalized Series
$\frac{1}{n}\cdot\cos\left[i\cdot(n-1)\cdot\left(\pi - \frac{\pi}{2}\right)\right] =$	$\left[\frac{1}{n}\cdot\cos\left[i\cdot(n-1)\cdot\left(\pi - \frac{\pi}{2}\right)\right]\right]^{-1} =$
1	1
0	
-0.3333	-3
0	
0.2	5
0	
-0.1429	-7
0	

$\frac{\pi}{4} = \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) = \sum_{k=1}^{\infty}\frac{(-1)^{2k-1}}{k}$

(No new information)

Evaluation at π

Real Space	Harmonic Space
$\frac{\left(\ln\left(1 + e^{i\cdot\pi}\right)\right)\cdot e^{-i\cdot\pi} + \left(\ln\left(1 + e^{-i\cdot\pi}\right)\right)\cdot e^{i\cdot\pi}}{2}$	$\sum_{n=1}^{\infty}\left[\frac{1}{n}\cdot\cos\left[i\cdot(n-1)\cdot(\pi - \pi)\right]\right]$
∞	$\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right)$

A normalization factor is not required in order to obtain the simple series.

Normalized Simple Series	Reciprocal of Normalized Series
$\frac{1}{n}\cdot\cos\left[i\cdot(n-1)\cdot(\pi - \pi)\right] =$	$\left[\frac{1}{n}\cdot\cos\left[i\cdot(n-1)\cdot(\pi - \pi)\right]\right]^{-1} =$
1	1
0.5	2
0.3333	3
0.25	4
0.2	5
0.1667	6
0.1429	7
0.125	8

$\infty = \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right) = \sum_{k=1}^{\infty}\frac{1}{k}$

(Does not Converge)