Casey's Curves

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XI
continued

Summary of Some of the Simple Harmonics

FIRST ORDER (PI RATIOS) continued from previous sheet

$$\frac{\theta}{2} = \sum_{n=1}^{\infty}\left[\frac{2}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\right] \quad \text{(Transform)}$$

Normalized Simple Series (First Piece)	Reciprocal of Normalized Series (First Piece)	Normalized Simple Series (Second Piece)	Reciprocal of Normalized Series (Second Piece)
$\frac{1}{\sqrt{2}}\cdot\frac{2}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{4}\right)\right]$ =	$\left[\frac{1}{\sqrt{2}}\cdot\frac{2}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{4}\right)\right]\right]^{-1}$ =	$\frac{2}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{4}\right)\right]$ =	$\left[\frac{2}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{4}\right)\right]\right]^{-1}$ =
Evaluation at π/2			
1	1	1.4142	0.7071
-0.7071	-1.4142	-1	-1
0.3333	3	0.4714	2.1213
0		0	
-0.2	-5	-0.2828	-3.5355
0.2357	4.2426	0.3333	3
-0.1429	-7	-0.202	
0		0	

$\frac{\pi}{4} = \sqrt{2}\left(1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\ldots\right)\cdot\left(1-\frac{1}{3}+\frac{1}{5}-\ldots\right)$

$\frac{\pi}{4} = \sum_{k=1}^{\infty}\left[\left[\sqrt{2}\cdot\left[\frac{(-1)^{k-1}}{4k-3}+\frac{(-1)^{k-1}}{4k-1}\right]\right]\cdot\left[\frac{(-1)^{k-1}}{4k-3}+\frac{(-1)^{k-1}}{(2k-1)}\right]\right]$ $\frac{\pi}{4}=0.7854$ $\sum_{k=1}^{10000}\left[\left[\sqrt{2}\cdot\left[\frac{(-1)^{k-1}}{4k-3}+\frac{(-1)^{k-1}}{4k-1}\right]\right]\cdot\left[\frac{(-1)^{k-1}}{4k-3}+\frac{(-1)^{k-1}}{(2k-1)}\right]\right]=0.7854$

If the Real and Harmonic Space maps are graphically equivalent then their derivatives and integrals must be be too; the slope of the two mapping functions, and likewise the area under the two mapping functions, must be graphically equivalent if the original maps were:

If $f(\theta) \underset{ge}{=} F(\Theta)$ then $\frac{d}{d\theta}f(\theta) \underset{ge}{=} \frac{d}{d\Theta}F(\Theta)$ and $\int f(\theta)\,d\theta \underset{ge}{=} \int F(\Theta)\,d\Theta$

If Thus successive integrations can be performed to determine the higher order pi ratios from the first order relation. The method of successive integration is shown in a later section

In this first example the first order real space function is linear, that is $f(\theta)=\theta$. In the next example the first order real space function is a constant, specifically $f(\theta)=\pi$. Since the Real and Harmonic Space maps are graphically equivalent then their integrals must be as well. This relation was used to determine the higher orders.

SECOND ORDER (PI SQUARED RATIOS) continued from previous sheet

$$\theta^2 = \sum_{n=1}^{\infty}\left[\frac{8}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\cdot\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\Theta}{2}\right)\right]\right]\right] \quad \text{(Transform)}$$

Normalized Simple Series	Reciprocal of Normalized Series
$2\cdot\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right]\cdot\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{2}\right)\right]\right]$ =	$\left[2\cdot\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right]\cdot\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{2}\right)\right]\right]\right]^{-1}$ =
Evaluation at π/2	
1	1
-0.5	-2
0.1111	9
0	
0.04	25
-0.0556	-18
0.0204	49
0	

$\left(\frac{\pi}{2}\right)^2 = 8\times\frac{1}{2}\left[\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)-\frac{1}{2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)\right] = 4\left[\left[\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\right]-\frac{1}{2}\left[\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\right]\right] = 2\left[\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\right]$

which simplies to $\frac{\pi^2}{8}=\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$ (No new information)

Since $\frac{\pi^2}{8}=\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)$ and $\frac{\pi^2}{12}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}=\left(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\ldots\right)$

then $\frac{\pi^2}{8}-\frac{\pi^2}{12}=\left(\frac{1}{2^2}+\frac{1}{4^2}+\ldots\right)=\sum_{k=1}^{\infty}\frac{1}{(2k)^2}=\frac{1}{4}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)$ which simplies to $\frac{\pi^2}{6}=\sum_{k=1}^{\infty}\frac{1}{k^2}$

$\frac{\pi^2}{6}=1.6449$ $\sum_{k=1}^{10000}\frac{1}{k^2}=1.6448$

THIRD ORDER (PI CUBED RATIOS) continued from previous sheet

$$\theta^3 = \sum_{n=1}^{\infty}\left[\frac{32}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right]\cdot\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]^2\right] \quad \text{(Transform)}$$

Normalized Simple Series (First Piece)	Normalized Simple Series (Second Piece)	Normalized Simple Series (Third Piece)
$\frac{1}{32\cdot\sqrt{2}\cdot\left(\sin\left(\frac{\pi}{8}\right)\right)^2}\cdot\frac{32}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right]\cdot\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]^2$ =	$\left(\frac{1}{4}\right)\cdot\frac{32}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right]\cdot\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]^2$ =	$\frac{27}{4\cdot\sqrt{2}\cdot\left(\sin\left(\frac{3\pi}{8}\right)\right)^2}\cdot\frac{32}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right]\cdot\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]^2$ =
Evaluation at π/2			
$\left(\frac{\pi}{2}\right)^2 = 32\cdot\sqrt{2}\cdot\left(\sin\left(\frac{\pi}{8}\right)\right)^2 \times$			
1	1.65685	37.05974
-0.60355	-1	-22.36753
0.21587	0.35766	8
0	0	0
-0.04663	-0.07725	-1.728
0.02235	0.03704	0.82843
-0.00292	-0.00483	-0.10805
0	0	0
Reciprocal	Reciprocal	Reciprocal
$\left[\frac{1}{32\cdot\sqrt{2}\cdot\left(\sin\left(\frac{\pi}{8}\right)\right)^2}\cdot\frac{32}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right]\cdot\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]^2\right]^{-1}$ =	$\left[\left(\frac{1}{4}\right)\cdot\frac{32}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right]\cdot\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]^2\right]^{-1}$ =	$\left[\frac{1}{32\cdot\sqrt{2}\cdot\left(\sin\left(\frac{3\pi}{8}\right)\right)^2}\cdot\frac{32}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right]\cdot\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]^2\right]^{-1}$ =
1	0.60355	5.82843
-1.65685	-1	-9.65685
4.63247	2.79594	27
-21.44661	-12.94417	-125
44.73506	27	260.73506
-343	-207.01881	-1999.1505

$\frac{\pi^3}{8} = 32\cdot\sqrt{2}\cdot\left(\sin\left(\frac{\pi}{8}\right)\right)^2\cdot\left(1-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{15^3}+\ldots\right)-4\cdot\left(1-\frac{1}{3^3}+\frac{1}{5^3}-\ldots\right)+32\cdot\sqrt{2}\cdot\left(\sin\left(\frac{3\pi}{8}\right)\right)^2\cdot\left(\frac{1}{3^3}-\frac{1}{5^3}+\frac{1}{11^3}-\frac{1}{13^3}+\ldots\right)$

so $\frac{\pi^3}{64} = \sum_{k=1}^{\infty}\left[4\cdot\sqrt{2}\cdot\left(\sin\left(\frac{\pi}{8}\right)\right)^2\cdot\left[\frac{1}{(8k-7)^3}-\frac{1}{(8k-1)^3}\right]-\frac{1}{2}\left[\frac{(-1)^{k-1}}{(2k-1)^3}\right]+4\cdot\sqrt{2}\cdot\left(\sin\left(\frac{3\pi}{8}\right)\right)^2\cdot\left[\frac{1}{(8k-5)^3}-\frac{1}{(8k-3)^3}\right]\right]$

But $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^3}=\frac{\pi^3}{32}$ so $\frac{\pi^3}{64}+\left(\frac{1}{2}\right)\cdot\frac{\pi^3}{32}=\frac{\pi^3}{32}=\sum_{k=1}^{\infty}\left[4\cdot\sqrt{2}\cdot\left(\sin\left(\frac{\pi}{8}\right)\right)^2\cdot\left[\frac{1}{(8k-7)^3}-\frac{1}{(8k-1)^3}\right]+4\cdot\sqrt{2}\cdot\left(\sin\left(\frac{3\pi}{8}\right)\right)^2\cdot\left[\frac{1}{(8k-5)^3}-\frac{1}{(8k-3)^3}\right]\right]$

or $\left[\frac{\pi^3}{128\cdot\sqrt{2}}=\sum_{k=1}^{\infty}\left[\left[\sin\left(\frac{\pi}{8}\right)\right]^2\cdot\left[\frac{1}{(8k-7)^3}-\frac{1}{(8k-1)^3}\right]+\left[\sin\left(\frac{3\pi}{8}\right)\right]^2\cdot\left[\frac{1}{(8k-5)^3}-\frac{1}{(8k-3)^3}\right]\right]\right]$ $\frac{\pi^3}{128\cdot\sqrt{2}}=0.1713$ $\sum_{k=1}^{10000}\left[\left[\sin\left(\frac{\pi}{8}\right)\right]^2\cdot\left[\frac{1}{(8k-7)^3}-\frac{1}{(8k-1)^3}\right]+\left[\sin\left(\frac{3\pi}{8}\right)\right]^2\cdot\left[\frac{1}{(8k-5)^3}-\frac{1}{(8k-3)^3}\right]\right]=0.1713$

XII

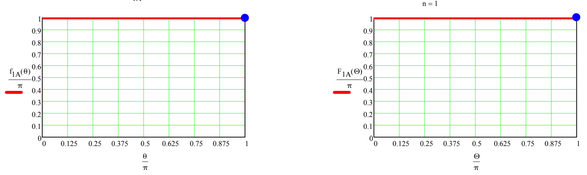
Another Simple Harmonic Series with Higher Orders Obtainable through Integration (this one has a constant 1st order base)

FIRST ORDER (CONSTANT BASED)

These maps are constant with both θ and Θ = π.

Real Space: $f_{1A}(\theta):=\pi$

Harmonic Space: $F_{1A}(\Theta):=\sum_{n=1}^{1000}\left[\frac{-8}{n\cdot i}\cdot\cosh\left[n\cdot\left(\frac{\Theta}{2}-\pi\right)\right]\cdot\sinh\left(\frac{i\cdot n\pi}{4}\right)\right]$



$\theta = \sum_{n=1}^{\infty}\left[\frac{-8}{n\cdot i}\cdot\cosh\left[n\cdot\left(\frac{\Theta}{2}-\pi\right)\right]\cdot\sinh\left(\frac{i\cdot n\pi}{4}\right)\right]$ (Transform)

(Real) (Harmonic)

Normalized Simple Series	Reciprocal of Normalized Series
$\frac{1}{4}\cdot\frac{-8}{n\cdot i}\cdot\cosh\left[n\cdot\left(\frac{\pi}{2}-\pi\right)\right]\cdot\sinh\left(\frac{i\cdot n\pi}{4}\right)$ =	$\left[\frac{1}{4}\cdot\frac{-8}{n\cdot i}\cdot\cosh\left[n\cdot\left(\frac{\pi}{2}-\pi\right)\right]\cdot\sinh\left(\frac{i\cdot n\pi}{4}\right)\right]^{-1}$ =
Evaluation at π	
0	
1	1
0	
0	
0	
-0.3333	-3
0	
0	

$\pi = 4\times$

$\pi = 4\cdot\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right)$ $\frac{\pi}{4}=\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)}$ (No new information)

Now use algebra to separate components.

From top of page $\frac{\pi}{4}=\sum_{k=1}^{\infty}\left[\left[\sqrt{2}\cdot\left[\frac{(-1)^{k-1}}{4k-3}+\frac{(-1)^{k-1}}{4k-1}\right]\right]-\frac{(-1)^{k-1}}{(2k-1)}\right]$ therefore $\frac{\pi}{2}=\sum_{k=1}^{\infty}\left[\sqrt{2}\cdot\left[\frac{(-1)^{k-1}}{4k-3}+\frac{(-1)^{k-1}}{4k-1}\right]\right]$

So $\frac{\pi}{2\cdot\sqrt{2}}=\left(1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\ldots\right)=\sum_{k=1}^{\infty}\left(\frac{1}{8k-7}+\frac{1}{8k-5}-\frac{1}{8k-3}-\frac{1}{8k-1}\right)=\sum_{k=1}^{\infty}\left[\frac{(-1)^{k-1}}{4k-3}+\frac{(-1)^{k-1}}{4k-1}\right]$

$\frac{\pi}{2\cdot\sqrt{2}}=1.1107$ $\sum_{k=1}^{10000}\left(\frac{1}{8k-7}+\frac{1}{8k-5}-\frac{1}{8k-3}-\frac{1}{8k-1}\right)=1.1107$ $\left[\sum_{k=1}^{10000}\left[\frac{(-1)^{k-1}}{4k-3}+\frac{(-1)^{k-1}}{4k-1}\right]\right]=1.1107$

Using $\frac{\pi}{4}=\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right)$ and $\frac{\pi}{2\cdot\sqrt{2}}=\left(1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\ldots\right)$ we see that

$\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right)+\left(1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\ldots\right)=2\left(1+0-0-\frac{1}{7}+\ldots\right)$ and $\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right)-\left(1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\ldots\right)=-2\left(0+\frac{1}{3}-\frac{1}{5}-0+\ldots\right)$

So $\frac{\pi}{4}+\frac{\pi}{2\cdot\sqrt{2}}=2\left[\sum_{k=1}^{\infty}\left(\frac{1}{8k-7}-\frac{1}{8k-1}\right)\right]$ and $\frac{\pi}{4}-\frac{\pi}{2\cdot\sqrt{2}}=-2\left[\sum_{k=1}^{\infty}\left(\frac{1}{8k-5}-\frac{1}{8k-3}\right)\right]$

Or $\left[\frac{\pi}{8}\cdot(1+\sqrt{2})=\sum_{k=1}^{\infty}\left(\frac{1}{8k-7}-\frac{1}{8k-1}\right)=\left(1-\frac{1}{7}+\frac{1}{9}-\frac{1}{15}+\ldots\right)\right]$ and $\left[\frac{\pi}{8}\cdot(\sqrt{2}-1)=\sum_{k=1}^{\infty}\left(\frac{1}{8k-5}-\frac{1}{8k-3}\right)=\left(\frac{1}{3}-\frac{1}{5}+\frac{1}{11}-\frac{1}{13}+\ldots\right)\right]$

$\frac{\pi}{8}(1+\sqrt{2})=0.9481$ $\sum_{k=1}^{10000}\left(\frac{1}{8k-7}-\frac{1}{8k-1}\right)=0.9481$ $\frac{\pi}{8}(\sqrt{2}-1)=0.1627$ $\sum_{k=1}^{10000}\left(\frac{1}{8k-5}-\frac{1}{8k-3}\right)=0.1627$

Combinations of Odd Terms that Converge to a Closed Form

The alternating series of odd terms $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2n-1}=\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right)$ can be separated into the two alternating series $\sum_{k=1}^{\infty}\left(\frac{1}{8k-7}-\frac{1}{8k-1}\right)=\left(1-\frac{1}{7}+\frac{1}{9}\right)$ and

$\sum_{k=1}^{\infty}\left(\frac{1}{8k-5}-\frac{1}{8k-3}\right)=\left(\frac{1}{3}-\frac{1}{5}+\frac{1}{11}\right)$. These both converge to closed forms. The invidual components: $\sum_{k=1}^{\infty}\frac{1}{8k-1}=\left(\frac{1}{7}+\frac{1}{15}+\frac{1}{23}\ldots\right)$ do not converge which can

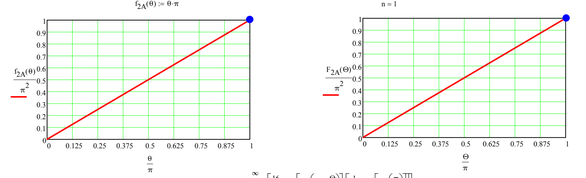
be shown by the Integral Test; we write the series as a function and evaluate the upper boundary of the integral, since it is infinite the series diverges: $\int^{\infty}\frac{1}{8n-1}\,dn=\frac{1}{8}\ln(8n-1)\Big|^{\infty}=\infty$

SECOND ORDER (CONSTANT BASED)

These maps are linear with θ and Θ.

Real Space: $f_{2A}(\theta):=\theta\cdot\pi$

Harmonic Space: $F_{2A}(\Theta):=\sum_{n=1}^{1000}\left[\frac{16}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\cdot\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]\right]$



$\theta\cdot\pi = \sum_{n=1}^{\infty}\left[\frac{16}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\cdot\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]\right]$ (Transform)

(Real) (Harmonic)

Normalized Simple Series	Reciprocal of Normalized Series
$\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right]\cdot\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]$ =	$\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right]\cdot\left[\frac{1}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]\right]^{-1}$ =
Evaluation at π	
1	1
0	
-0.1111	-9
0	
-0.04	-25
0	
0.0204	49
0	

$\pi^2 = \frac{16}{\sqrt{2}}\left(1-\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right)$

$\frac{\pi^2}{8\cdot\sqrt{2}}=\left(1-\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right)=\sum_{k=1}^{\infty}\left[\frac{1}{(8k-7)^2}-\frac{1}{(8k-5)^2}-\frac{1}{(8k-3)^2}+\frac{1}{(8k-1)^2}\right]=\sum_{k=1}^{\infty}\left[\frac{(-1)^{k-1}}{(4k-3)^2}-\frac{(-1)^{k-1}}{(4k-1)^2}\right]$

$\frac{\pi^2}{8\cdot\sqrt{2}}=0.8724$ $\sum_{k=1}^{10000}\left[\frac{1}{(8k-7)^2}-\frac{1}{(8k-5)^2}-\frac{1}{(8k-3)^2}+\frac{1}{(8k-1)^2}\right]=0.8724$ $\sum_{k=1}^{10000}\left[\frac{(-1)^{k-1}}{(4k-3)^2}-\frac{(-1)^{k-1}}{(4k-1)^2}\right]=0.8724$

Since $\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right)=\frac{\pi^2}{8}$ then

$\frac{\pi^2}{8}+\frac{\pi^2}{8\cdot\sqrt{2}}=\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right)+\left(1-\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right)=2\left(1-0-0+\frac{1}{7^2}+\ldots\right)$

Or $\left[\left(1+\frac{1}{\sqrt{2}}\right)\cdot\frac{\pi^2}{16}=\left(1+\frac{1}{7^2}+\ldots\right)=\sum_{k=1}^{\infty}\left[\frac{1}{(8k-7)^2}+\frac{1}{(8k-1)^2}\right]\right]$

Also $\frac{\pi^2}{8}-\frac{\pi^2}{8\cdot\sqrt{2}}=\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right)-\left(1-\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right)=2\left(0+\frac{1}{3^2}+\frac{1}{5^2}+0+\ldots\right)$

So $\left[\left(1-\frac{1}{\sqrt{2}}\right)\cdot\frac{\pi^2}{16}=\left(\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)=\sum_{k=1}^{\infty}\left[\frac{1}{(8k-5)^2}+\frac{1}{(8k-3)^2}\right]\right]$

$\left(1+\frac{1}{\sqrt{2}}\right)\cdot\frac{\pi^2}{16}=1.053$ $\sum_{k=1}^{10000}\left[\frac{1}{(8k-7)^2}+\frac{1}{(8k-1)^2}\right]=1.053$ $\left(1-\frac{1}{\sqrt{2}}\right)\cdot\frac{\pi^2}{16}=0.1807$ $\sum_{k=1}^{10000}\left[\frac{1}{(8k-5)^2}+\frac{1}{(8k-3)^2}\right]=0.1807$

The series of odd terms $\sum_{k=1}^{\infty}\frac{1}{(2n-1)^2}=\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right)$ can be separated into $\sum_{k=1}^{\infty}\left[\frac{1}{(8k-7)^2}+\frac{1}{(8k-1)^2}\right]=\left(1+\frac{1}{7^2}+\frac{1}{9^2}\ldots\right)$ and

$\sum_{k=1}^{\infty}\left[\frac{1}{(8k-5)^2}+\frac{1}{(8k-3)^2}\right]=\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{11^2}\ldots\right)$. Both of these added series combinations converge to closed forms. The invidual components such as

$\sum_{k=1}^{\infty}\frac{1}{(8k-1)^2}=\left(\frac{1}{7^2}+\frac{1}{15^2}+\frac{1}{23^2}\ldots\right)$ converge which can be shownby the Integral Test. Since the integral of the function is finite the series converges:

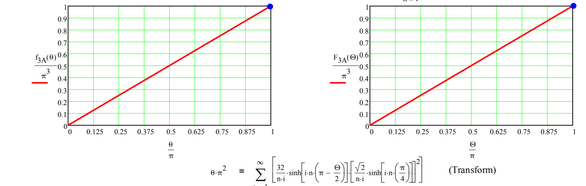
$\int^{\infty}\frac{1}{(8n-1)^2}\,dn=\frac{-1}{8n-1}\Big|^{\infty}=1$. If the indidual components were separable to closeds forms then we could algebraicly rearrange them to create a closed form for $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2}$

THIRD ORDER (CONSTANT BASED)

These maps are linear with θ and Θ.

Real Space: $f_{3A}(\theta):=\theta\cdot\pi^2$

Harmonic Space: $F_{3A}(\Theta):=\sum_{n=1}^{1000}\left[\frac{64}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\cdot\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]^2\right]$



$\theta\cdot\pi^2 = \sum_{n=1}^{\infty}\left[\frac{32}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\cdot\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]^2\right]$ (Transform)

(Real) (Harmonic)

Normalized Simple Series (First Piece)	Reciprocal of Normalized Series (First Piece)	Normalized Simple Series (Second Piece)	Reciprocal of Normalized Series (Second Piece)
$\frac{\sqrt{2}}{64}\cdot\frac{64}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{4}\right)\right]\cdot\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]^2$ =	$\left[\frac{\sqrt{2}}{64}\cdot\frac{64}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right]\cdot\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]^2\right]^{-1}$ =	$\frac{1}{128}\cdot\frac{64}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right]\cdot\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]^2$ =	$\left[\frac{1}{128}\cdot\frac{64}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)\right]\cdot\left[\frac{\sqrt{2}}{n\cdot i}\cdot\sinh\left[i\cdot n\cdot\left(\frac{\pi}{4}\right)\right]\right]^2\right]^{-1}$ =
Evaluation at π			
$\pi^3 = \frac{64}{\sqrt{2}}\times$			
1	1	0.35355	2.82843
-0.35355	-2.82843	-0.125	-8
0.03704	27	0.01309	76.36753
0		0	
-0.008	-125	-0.00283	-353.55339
0.01309	76.36753	0.00463	216
-0.00292	-343	-0.00103	-970.1505
0		0	

$\pi^3 = \frac{64}{\sqrt{2}}\left(1+\frac{1}{3^3}-\frac{1}{5^3}-\frac{1}{7^3}+\ldots\right)-128\left(\frac{1}{3^3}-\frac{1}{6^3}+\ldots\right)$ $\left[\frac{\sqrt{2}}{64}\cdot\pi^3=\sum_{k=1}^{\infty}\left[\frac{1}{(8k-7)^3}-\frac{2\cdot\sqrt{2}}{(8k-6)^3}+\frac{1}{(8k-5)^3}-\frac{1}{(8k-3)^3}+\frac{2\cdot\sqrt{2}}{(8k-2)^3}-\frac{1}{(8k-1)^3}\right]\right]$

$\frac{\sqrt{2}}{64}\cdot\pi^3=0.6851$ $\sum_{k=1}^{10000}\left[\frac{1}{(8k-7)^3}-\frac{2\cdot\sqrt{2}}{(8k-6)^3}+\frac{1}{(8k-5)^3}-\frac{1}{(8k-3)^3}+\frac{2\cdot\sqrt{2}}{(8k-2)^3}-\frac{1}{(8k-1)^3}\right]=0.6851$

But $\sum_{k=1}^{\infty}\left[\frac{1}{(8k-6)^3}-\frac{1}{(8k-2)^3}\right]=\frac{1}{8}\left[\sum_{k=1}^{\infty}\left[\frac{1}{(4k-3)^3}-\frac{1}{(4k-1)^3}\right]\right]=\frac{1}{8}\cdot\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^3}=\frac{1}{8}\cdot\frac{\pi^3}{32}=\frac{\pi^3}{256}$

So $\frac{\sqrt{2}}{64}\cdot\pi^3+2\cdot\sqrt{2}\cdot\frac{\pi^3}{256}=\frac{3\cdot\sqrt{2}}{128}\cdot\pi^3=\sum_{k=1}^{\infty}\left[\frac{1}{(8k-7)^3}+\frac{1}{(8k-5)^3}-\frac{1}{(8k-3)^3}-\frac{1}{(8k-1)^3}\right]=\sum_{k=1}^{\infty}\left[\frac{(-1)^{k-1}}{(4k-3)^3}+\frac{(-1)^{k-1}}{(4k-1)^3}\right]$ (1)

$\frac{3\cdot\sqrt{2}}{128}\cdot\pi^3=1.0277$ $\sum_{k=1}^{10000}\left[\frac{1}{(8k-7)^3}+\frac{1}{(8k-5)^3}-\frac{1}{(8k-3)^3}-\frac{1}{(8k-1)^3}\right]=1.0277$ $\sum_{k=1}^{10000}\left[\frac{(-1)^{k-1}}{(4k-3)^3}+\frac{(-1)^{k-1}}{(4k-1)^3}\right]=1.0277$

Also $\frac{\pi^3}{32}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^3}=\sum_{k=1}^{\infty}\left[\frac{1}{(8k-7)^3}-\frac{1}{(8k-5)^3}+\frac{1}{(8k-3)^3}-\frac{1}{(8k-1)^3}\right]$ (2)

By adding (1) + (2) and dividing by 2 $\frac{\pi^3}{32}+\frac{3\cdot\sqrt{2}}{128}\cdot\pi^3=\left[\left(1+\frac{3\cdot\sqrt{2}}{4}\right)\cdot\frac{\pi^3}{64}=\sum_{k=1}^{\infty}\left[\frac{1}{(8k-7)^3}-\frac{1}{(8k-1)^3}\right]\right]$

By subtracting (1) - (2) and dividing by 2 $\frac{\pi^3}{32}-\frac{3\cdot\sqrt{2}}{128}\cdot\pi^3=\left[\left(\frac{3\cdot\sqrt{2}}{4}-1\right)\cdot\frac{\pi^3}{64}=\sum_{k=1}^{\infty}\left[\frac{1}{(8k-5)^3}-\frac{1}{(8k-3)^3}\right]\right]$

$\left(1+\frac{3\cdot\sqrt{2}}{4}\right)\cdot\frac{\pi^3}{64}=0.9983$ $\sum_{k=1}^{10000}\left[\frac{1}{(8k-7)^3}-\frac{1}{(8k-1)^3}\right]=0.9983$ $\left(\frac{3\cdot\sqrt{2}}{4}-1\right)\cdot\frac{\pi^3}{64}=0.0294$ $\sum_{k=1}^{10000}\left[\frac{1}{(8k-5)^3}-\frac{1}{(8k-3)^3}\right]=0.0294$

The alternating series of odd terms $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2n-1)^3}=\left(1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}-\ldots\right)$ can be separated into the two alternating series $\sum_{k=1}^{\infty}\left[\frac{1}{(8k-7)^3}-\frac{1}{(8k-1)^3}\right]=\left(1-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{15^3}\ldots\right)$ and

$\sum_{k=1}^{\infty}\left[\frac{1}{(8k-5)^3}-\frac{1}{(8k-3)^3}\right]=\left(\frac{1}{3^3}-\frac{1}{5^3}+\frac{1}{11^3}\right)$. These both converge to closed forms. The invidual components such as $\sum_{k=1}^{\infty}\frac{1}{(8k-7)^3}=\left(1+\frac{1}{9^3}+\frac{1}{17^3}\ldots\right)$ converge which can

be shown by the Integral Test. If we could write the individual components in closed form we could algebraicly rearrange these to create a closed forms for $\sum_{k=1}^{\infty}\frac{1}{(2