Casey's Curves - The Big Graph

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These are plots of the imaginary and real components of the function in Harmonic Space $F_p(\Theta) = \sum_{n=1}^{\infty} \frac{e^{i\cdot n\cdot(\pi-\Theta)}}{n^p}$ where p is the exponent of the polylog $Li_p(z) = \sum_k \frac{z^k}{k^p}$ as $F(\Theta)$, with $z = e^{i\cdot n(\pi-\Theta)}$ to shift the x-axis by π for display.

The y-axis is $F_p(\Theta) = \sum_{n=1}^{\infty} \frac{e^{i\cdot n\cdot(\pi-\Theta)}}{n^p}$. Each F_p is a sum of waves in harmonic space. The left graph is the imaginary component and the right graph is the real component. The x-axis is $\frac{\Theta}{\pi}$, $e^{i(\pi-\pi)}$ has values -1 ,+i, +1, -i where shown.

When we evaluate the polylog at these points each wave set collapses to an infinite series.

EXAMPLE Dilog(i)

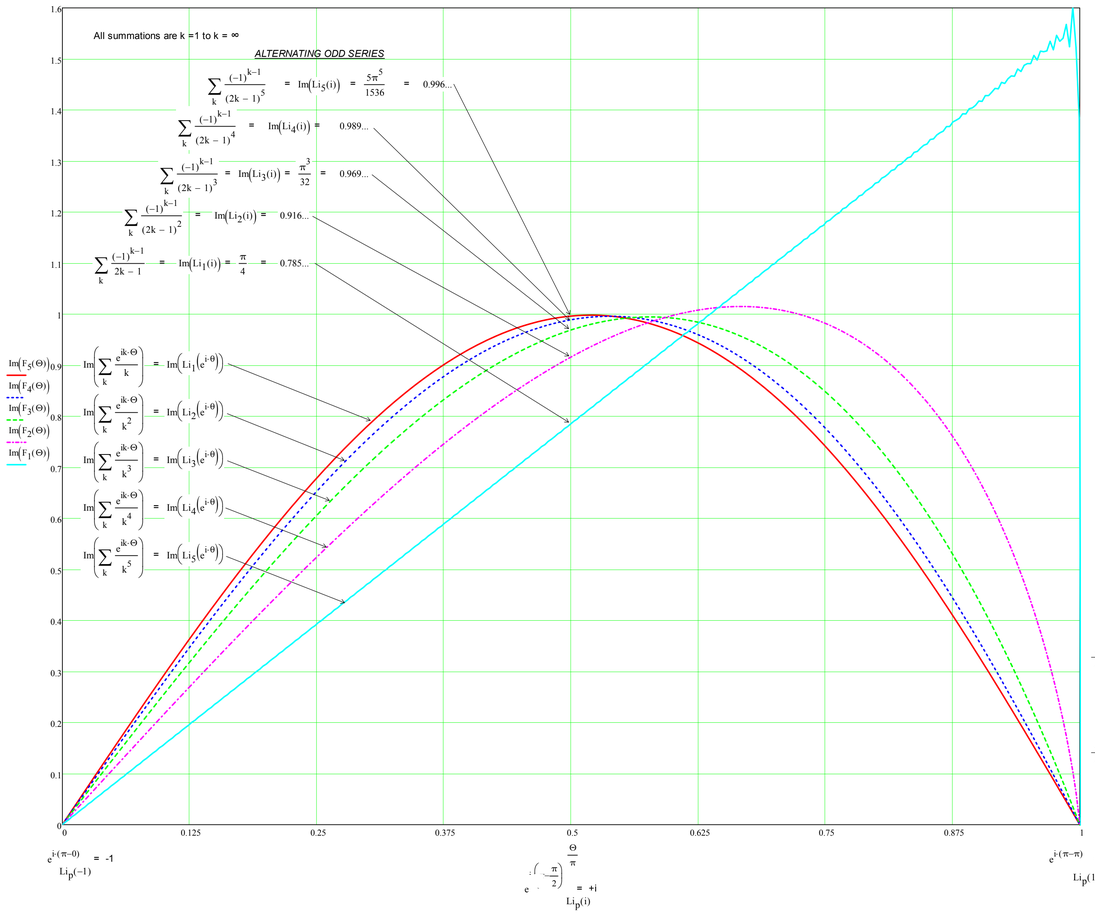
$\text{Dilog}(i) = Li_2(i) = \sum_n \frac{(i)^n}{n^2} =$ (1.0006, −0.250, −0.1111, 0.063, 0.0406, −0.028, −0.0208, 0.016)

$= i\left(1 - \frac{1}{9} + \frac{1}{25} - \ldots\right) + \frac{1}{4}\left(-1 + \frac{1}{4} - \frac{1}{9} + \ldots\right)$

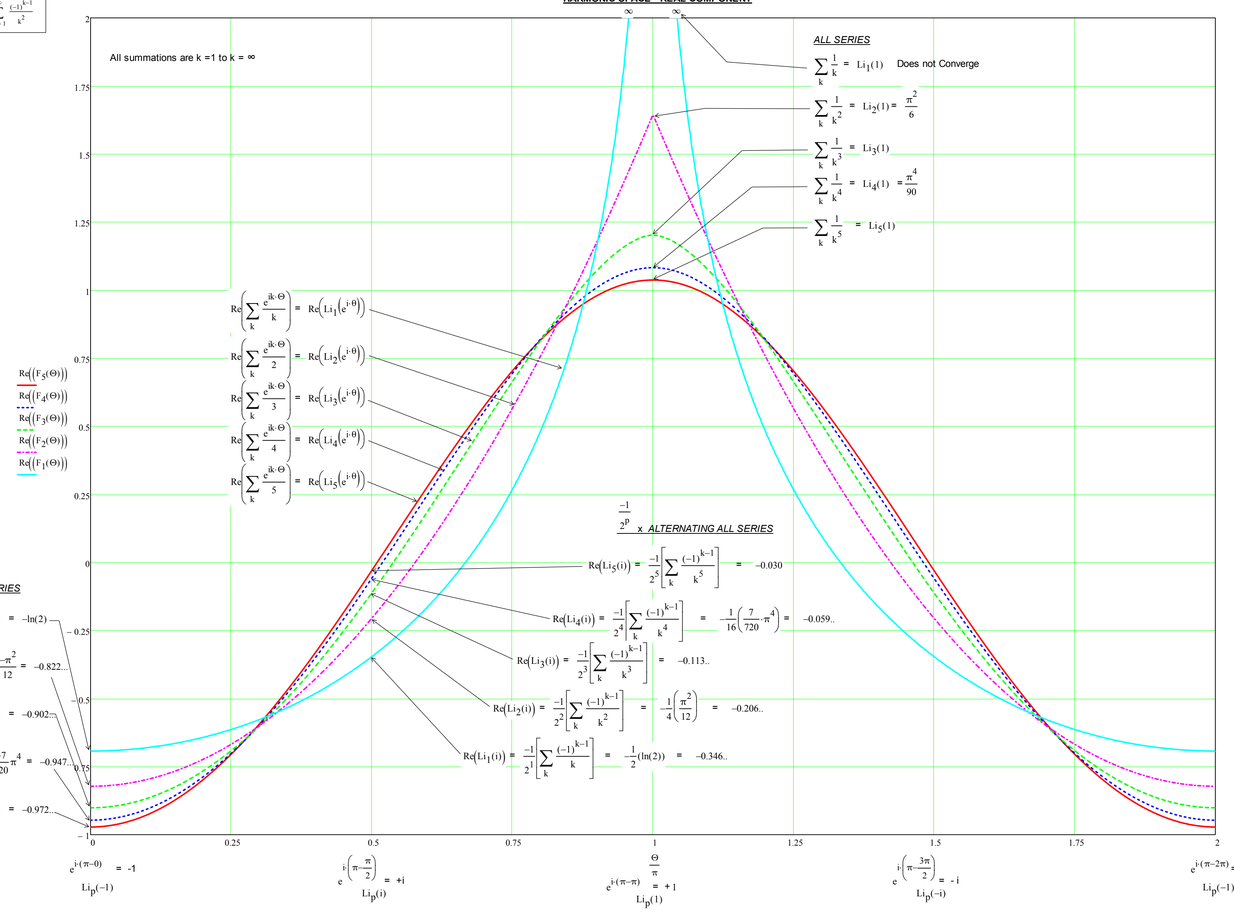
$= i\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2} + \frac{1}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}$

For example at $\Theta = \frac{\pi}{2}$, $e^{i\left(\pi-\frac{\Theta}{2}\right)} = +i$. For the dilog, p = 2 and $Li_2(i) = \sum_{k=1}^{\infty}\frac{(i)^k}{k^2} = i\cdot\left(1 - \frac{1}{9} + \frac{1}{25} - \ldots\right) + \frac{1}{4}\left(-1 + \frac{1}{4} - \frac{1}{9} + \ldots\right) = i\cdot\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2} + \frac{1}{2^2}\cdot\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}$. In general the imaginary component of the polylog at +i is equal to the *alternating over odd integer series:* $\text{Im}\left(Li_p(i)\right) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^p}$ where p is the power of the exponent or the order. The real part is equal to $\frac{1}{2^p}$ x *the alternating all integers series* $\text{Re}\left(Li_p(i)\right) = \frac{1}{2^p}\cdot\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^p}$.

HARMONIC SPACE - IMAGINARY COMPONENT



HARMONIC SPACE - REAL COMPONENT



ALTERNATING ALL SERIES

$-\sum_k \frac{(-1)^{k-1}}{k} = Li_1(-1) = -\ln(2)$

$-\sum_k \frac{(-1)^{k-1}}{k^2} = Li_2(-1) = \frac{-\pi^2}{12} = -0.822...$

$-\sum_k \frac{(-1)^{k-1}}{k^3} = Li_3(-1) = -0.902...$

$-\sum_k \frac{(-1)^{k-1}}{k^4} = Li_4(-1) = \frac{-7}{720}\pi^4 = -0.947...$

$-\sum_k \frac{(-1)^{k-1}}{k^5} = Li_5(-1) = -0.972...$

The polylog by definition also has a graphical equivalent in Real Space of $Li_1(z) = -\ln(1-z)$, $Li_{p+1}(z) = \int_0^z \frac{Li_p(t)}{t}dt$ where p is the order. We recognize that $\sum_k \frac{z}{k} = -\ln(1-z)$ at z = 1, $\sum_k \frac{z^k}{k^p} = \int_0^z \frac{Li_p(t)}{t}dt$ at z > 1 is a transform.

Some of the simpler real space forms are shown in the table.

ALTERNATING ODD SERIES

Infinite Series (Waveset Collapsed to One Pont)	Graph Point	Graph Point (simplified)	Polylog Summation Form	Closed Form	Decimal Form	Real Space Transform Form of Polylog
$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}$	$\text{Im}\left[Li_1\left[e^{i\left(\pi-\frac{\pi}{2}\right)}\right]\right]$	$\text{Im}(Li_1(i))$	$\text{Im}\left(\sum_k \frac{i^k}{k}\right)$	$\frac{\pi}{4}$	0.785...	$\frac{\ln(1+i)-\ln(1-i)}{2\cdot i}$
$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2}$	$\text{Im}\left[Li_2\left[e^{i\left(\pi-\frac{\pi}{2}\right)}\right]\right]$	$\text{Im}(Li_2(i))$	$\text{Im}\left(\sum_k \frac{i^k}{k^2}\right)$	$\frac{Li_2(i)-Li_3(-i)}{2\cdot i}$	0.916...	$\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\ln\left(1-e^{i\cdot\theta}\right)d\theta$
$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^3}$	$\text{Im}\left[Li_3\left[e^{i\left(\pi-\frac{\pi}{2}\right)}\right]\right]$	$\text{Im}(Li_3(i))$	$\text{Im}\left(\sum_k \frac{i^k}{k^3}\right)$	$\frac{\pi^3}{32}$	0.969...	$\frac{1}{8\cdot i}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi} -\ln\left(1-e^{i\cdot\theta}\right)d\theta\,d\theta$
$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^4}$	$\text{Im}\left[Li_4\left[e^{i\left(\pi-\frac{\pi}{2}\right)}\right]\right]$	$\text{Im}(Li_4(i))$	$\text{Im}\left(\sum_k \frac{i^k}{k^4}\right)$	$\frac{Li_4(i)-Li_4(-i)}{2\cdot i}$	0.989...	$\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta\,d\theta\,d\theta + \frac{1}{2}\cdot\pi\cdot C_3$
$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^5}$	$\text{Im}\left[Li_5\left[e^{i\left(\pi-\frac{\pi}{2}\right)}\right]\right]$	$\text{Im}(Li_5(i))$	$\text{Im}\left(\sum_k \frac{i^k}{k^5}\right)$	$\frac{5}{1536}\pi^5$	0.996...	$\frac{1}{32\cdot i}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\theta} -\ln\left(1-e^{i\cdot\theta}\right)d\theta\,d\theta\,d\theta\,d\theta$

ALTERNATING ALL SERIES

(negated from graph for simpliciy)

InfiniteSeries	Graph Point (simplified)	Other Graph Point (simplified)	Closed Form	Decimal Form	Real Space Transform
$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}$	$-Li_1(-1)$	$2^1\text{Re}(Li_1(i))$	$\ln(2)$	0.693...	$\ln(1+i)+\ln(1-i) = 0.693$
$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}$	$-Li_2(-1)$	$2^2\text{Re}(Li_2(i))$	$\frac{\pi^2}{12}$	0.822...	$\frac{1}{3\cdot i}\int_0^{\pi} -\ln\left(1-e^{i\cdot\theta}\right)d\theta$
$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3}$	$-Li_3(-1)$	$2^3\text{Re}(Li_3(i))$	$Li_3(-1) = 0.902...$	0.902...	$\frac{3}{14}\cdot\int_0^{\pi}\int_{-\theta}^{\theta} -\ln\left(1-e^{i\cdot\theta}\right)d\theta\,d\theta$
$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^4}$	$-Li_4(-1)$	$2^4\text{Re}(Li_4(i))$	$\frac{7}{720}\pi^4 = 0.947...$	0.947...	$\frac{7}{180\cdot i}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi} -\ln\left(1-e^{i\cdot\theta}\right)d\theta\,d\theta\,d\theta$
$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^5}$	$-Li_5(-1)$	$2^5\text{Re}(Li_5(i))$	$Li_5(-1) = 0.972...$	0.972...	$\frac{15}{31}\int_0^{\pi}\int_0^{\theta}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta + i\cdot\frac{\pi^2}{6}d\theta + C_3\,d\theta - \frac{\pi^4}{90}\cdot i\,d\theta$

$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^p} = \frac{2^{(p-1)}-1}{2^{(p-1)}}\cdot\sum_{k=1}^{\infty}\frac{1}{k}$

general formula can de shown algebraically

ALT ALL SERIES : ALL SERIES RATIO

ODD SERIES

InfiniteSeries	Ratio to All Series	Closed Form	Decimal Form	Real Space Transform
$\sum_{k=1}^{\infty}\frac{1}{2k-1}$	$\frac{1}{2}\cdot\sum_{k=1}^{\infty}\frac{1}{k}$	$\frac{1}{2}Li_1(1)$	∞	$\ln(0)$
$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$	$\frac{3}{4}\cdot\sum_{k=1}^{\infty}\frac{1}{k^2}$	$\frac{\pi^2}{8}$	1.234...	$\frac{1}{2\cdot i}\int_0^{\pi} -\ln\left(1-e^{i\cdot\theta}\right)d\theta$
$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}$	$\frac{7}{8}\cdot\sum_{k=1}^{\infty}\frac{1}{k^3}$	$\frac{7}{8}Li_3(1)$	1.0518...	$\frac{1}{4}\cdot\int_0^{\pi}\int_{-\theta}^{\theta} -\ln\left(1-e^{i\cdot\theta}\right)d\theta\,d\theta$
$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}$	$\frac{15}{16}\cdot\sum_{k=1}^{\infty}\frac{1}{k^4}$	$\frac{\pi^4}{96}$	1.0147...	$\frac{1}{24\cdot i}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi} -\ln\left(1-e^{i\cdot\theta}\right)d\theta\,d\theta\,d\theta$
$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^5}$	$\frac{31}{32}\cdot\sum_{k=1}^{\infty}\frac{1}{k^5}$	$\frac{31}{32}Li_5(1)$	1.0045..	$\frac{1}{2}\int_0^{\pi}\int_0^{\theta}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta\,d\theta + C_3\,d\theta\,d\theta + \frac{\pi^5}{120}\cdot i$

$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^p} = \frac{2^p-1}{2^p}\cdot\sum_{k=1}^{\infty}\frac{1}{k}$

general formula can de shown algebraically

ODD SERIES : ALL SERIES RATIO

ALL SERIES

InfiniteSeries	Graph Point (simplified)	Closed Form	Decimal Form	Real Space Transform
$\sum_{k=1}^{\infty}\frac{1}{k}$	$Li_1(1)$	$Li_1(1)$	∞	$\ln(0)$
$\sum_{k=1}^{\infty}\frac{1}{k^2}$	$Li_2(1)$	$\frac{\pi^2}{6}$	1.645...	$\frac{2}{3\cdot i}\int_0^{\pi} -\ln\left(1-e^{i\cdot\theta}\right)d\theta$
$\sum_{k=1}^{\infty}\frac{1}{k^3}$	$Li_3(1)$	$Li_3(1)$	1.202...	$\frac{2}{7}\cdot\int_0^{\pi}\int_{-\theta}^{\theta} -\ln\left(1-e^{i\cdot\theta}\right)d\theta\,d\theta = C_3$
$\sum_{k=1}^{\infty}\frac{1}{k^4}$	$Li_4(1)$	$\frac{\pi^4}{90}$	1.0823...	$\frac{8}{15}\int_0^{\pi}\int_0^{\theta}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta + i\cdot\frac{\pi^2}{6}d\theta + C_3\,d\theta + C_3\,d\theta$
$\sum_{k=1}^{\infty}\frac{1}{k^5}$	$Li_5(1)$	$Li_5(1)$	1.0369...	$\frac{16}{31}\int_0^{\pi}\int_0^{\theta}\int_0^{\theta}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta + i\cdot\frac{\pi^2}{6}d\theta + C_3\,d\theta + i\cdot\frac{\pi^4}{90}d\theta$