Part VI: A. Fibonacci Numbers

Here we derive the equation for the Fibonacci numbers and show how the mechanics works. It is assumed that the reader is familiar with the work on e and π so this discussion streamlined. The equation we will derive provides the n^th Fibonacci Number f_n :

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} \tag{60}$$

Each Fibonacci number is an integer and is the sum of the two Fibonacci numbers before it. Here nature is tasked with tracking two numbers and adding them together. Let us see how this is done.

We start by measuring an action in both real and harmonic space and require that the results be the same for both observers. This is our Equivalence Relation or General Transform Equation (GTE):

GTE $$\int_A^B f(x)dx = \int_A^B F(X)dX \tag{6.1}$$

We let $f(x) = f_n$, where f_n is the n^th Fibonacci number. We seek F(X) which satisfies Equation (6.1).

Let $F(X) = n \cdot X^{n-1}$ Then $$\int_A^B f_n \cdot dx = \int_A^B n \cdot X^{n-1} dX \tag{6.2}$$

The choice of the boundary conditions A and B is very important as this is what governs the tracking numbers for the series. *This implies that we could track other series with rule sets different than Fibonacci by using different values for A and B.* For now we state

$$A = (1-\sqrt{5})/2 \quad \text{and} \quad B = (1+\sqrt{5})/2 \tag{6.3}$$

which we will address later. So

$$\int_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} f_n dx = \int_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} n \cdot X^{n-1} dX \tag{6.4}$$

Evaluating the left hand side of (6.4):

$$\int_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} f_n dx = f_n \cdot x \Big|_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} = f_n \cdot \left(\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right) \right) = \sqrt{5} f_n \tag{6.5}$$

Evaluating the right hand side of (6.4):

$$\int_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} n \cdot X^{n-1} dX = \quad X^n \Big|_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} \tag{6.6}$$

Substituting (6.5) & (6.6) into (6.4):

$$\sqrt{5} f_n = X^n \Big|_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} \tag{6.7}$$

Therefore the nth Fibonacci integer must equal:

$$f_n = \quad \frac{1}{\sqrt{5}} \cdot \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right] = \quad \frac{\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n}{\sqrt{5}} \tag{6.8}$$

This is our desired result and we have completed the derivation of (6.0), with exception of the Boundary Conditions. Now we want to look at the mechanics of this.

B. Equivalence

The derivation of (6.8) came from the GTE using a similar procedure to that of e and π . Also, since each Fibonacci integer is the sum of the two Fibonacci integers before it, we can write

$$\int_A^B f_n \cdot dx = \quad \int_A^B F(X) dX = \quad \int_A^B F_{n-1}(X) dX + \int_A^B F_{n-2}(X) dX \tag{6.9}$$

where F_{n-1} and F_{n-2} are the harmonic transforms of the previous Fibonacci integer and the one before that, respectively.

When an action occurs in real space that requires the Fibonacci Numbers it is demanding a solution to Equation (6.9) which is a differential equation (DE) expressed in integral form. This is analogous to plucking a guitar string. The guitar string can only vibrate in certain modes determined by the DE. The action demands a solution to the DE and is subject to its mechanics. This is the method of implementation.

In this case the n^{th} Fibonacci Number, f_n , equals its harmonic transform per (6.8) so there is no need to show equivalence between these. f_n is determined by the difference of $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$ raised to increasing powers of *n*.

According to (6.9) f_n is equal to the sum of the integrated harmonic transforms of the two previous numbers F_{n-1} and F_{n-2}. This implies:

$$F_{n-1} = (n-1)\cdot X^{n-2} \quad \text{and} \quad F_{n-2} = (n-2)\cdot X^{n-3} \tag{6.10}$$

To satisfy (6.9) we let:

$$\sqrt{5} f_n = X^n \Big|_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} = X^{n-1} \Big|_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} + X^{n-2} \Big|_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} \tag{6.11}$$

And these are the two previous Fibonacci integers:

$$f_{n-1} = \frac{1}{\sqrt{5}} X^{n-1} \Big|_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} \qquad f_{n-2} = \frac{1}{\sqrt{5}} X^{n-2} \Big|_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} \tag{6.12}$$

So the harmonic transforms add in the same way that the Fibonacci Numbers do.

And in general the nth Fibonacci number is given by:

$$f_n = \frac{1}{\sqrt{5}} X^n \Big|_{(1-\sqrt{5})/2}^{(1+\sqrt{5})/2} \tag{6.13}$$

Equation (6.11) is used to demonstrate graphical equivalence. The n^{th} Fibonacci Number, f_n is equal to the sum of the integrated harmonic transforms of the two previous numbers F_{n-1} and F_{n-2}. See Figure 60.

$$n := 1, 2 .. 12 \qquad A := \frac{1-\sqrt{5}}{2} \qquad B := \frac{1+\sqrt{5}}{2}$$

$$f(n) := \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} \qquad F(n) := \frac{\left[B^{n-1} - \left(A^{n-1}\right) + B^{n-2} - A^{n-2}\right]}{\sqrt{5}}$$

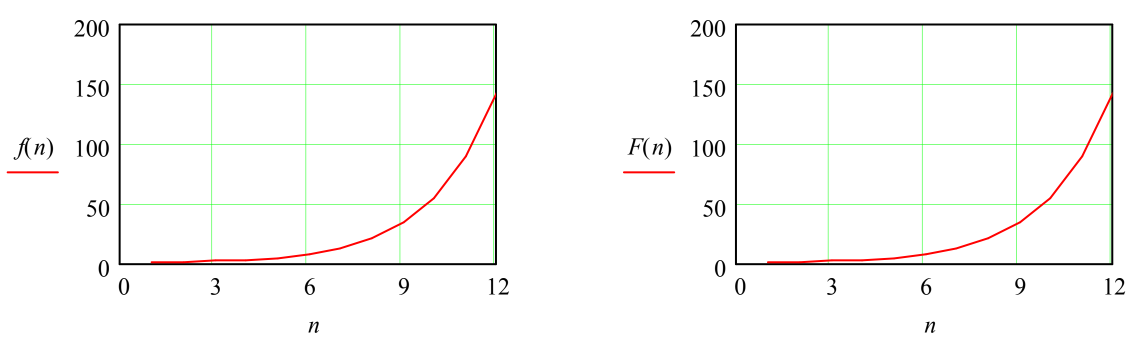

Figure 60 - The n^{th} Fibonacci Number, f_n is equal to the sum of the integrated harmonic transforms of the two previous numbers F_{n-1} and F_{n-2} as per Equation (6.11).

C. Binomial Expansion & Tracking Numbers

We now look at how the boundary conditions A and B govern the mechanics of the differential equation, enabling it to satisfy the requirement that the n^{th} Fibonacci number is the sum of the previous two values, and also that all of these values are integers.

Equation (6.13) is a remarkable equation. Not only does it tell us what the n^{th} Fibonacci number is, it also tells us and what to add to this to get the next Fibonacci number. It does this by tracking two numbers. One determines the current value and the other what to add to get the next value. It is able to do this by raising the mixed numbers $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$ to increasing powers of *n*. Under binomial expansion the rational and irrational portions combine separately, and there are two coefficients, one for the rational portion and one for the irrational portion. These are the two tracking numbers. When (6.13) requires us to take the difference of the two expansions, the irrational portions cancel out and we get a rational (integer) answer. In this way one of the tracking numbers is hidden and unobservable. This is another instance where there are insufficient variables in real space for the natural process. Here Nature needed a way to track a second value. The DE (6.9) is the method and the solution (6.13) contains the coding pattern. *In general the presence of rational and irrational numbers in a DE may indicate that unobservable values are being tracked.*

The information provided by (6.13) is a complete set meaning that by tracking these two numbers for each n, the entire set is determined. The evaluation at $n-1$ tells what needs to be added to reach that at n. That is equivalent to what needs to be subtracted from the value at n to get the value at $n-1$.

Now we examine the underlying mechanics. We return to Equation (6.0)

$$f_n = \frac{1}{\sqrt{5}} \cdot \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right] \tag{6.0}$$

At each *n*, there is a binomial expansion of the complex terms. For $n=2$:

$$\left(\frac{1+1\sqrt{5}}{2} \right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{3+1\sqrt{5}}{2} \quad \text{and} \quad \left(\frac{1-1\sqrt{5}}{2} \right)^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{3-1\sqrt{5}}{2} \tag{6.14}$$

The real and irrational parts of the expansion do not mix. The real numbers add to real numbers and the irrational numbers add to irrational numbers. In this way two separate properties can be controlled by a single mixed number.

As we will show, only one of these numbers is physically observable. The unobservable number is "dormant" in the physical process. Note that the binomial expansion results in a term where the irrational component of base $\sqrt{5}$ multiplies by another component of base $\sqrt{5}$ to produce a real number of base 5 which then adds in to the real numbers. In this manner the unobservable or dormant term plays a part in the physical process but is not visible. It contains information on how to calculate the value at future cycles where each n is a cycle. Figure 61 shows the binomial expansion for $n=1$ to $n=6$. Note that for each n there are two separate coefficients, one for the rational portion and one for the irrational portion. Such a property is imperative to natural phenomena such as growth. A plant may currently have three leaves, but that condition must also contain the information on the number of leaves in the next growth cycle, and that one must contain the information for the one which follows.

n=1	$\left(\frac{1+\sqrt{5}}{2}\right)^1 = \frac{1+\sqrt{5}}{2}$		$\left(\frac{1-\sqrt{5}}{2}\right)^1 = \frac{1-\sqrt{5}}{2}$
n=2	$\left(\frac{1+\sqrt{5}}{2}\right)\cdot\left(\frac{1+\sqrt{5}}{2}\right) = \frac{3+\sqrt{5}}{2}$		$\left(\frac{1-\sqrt{5}}{2}\right)\cdot\left(\frac{1-\sqrt{5}}{2}\right) = \frac{3-\sqrt{5}}{2}$
n=3	$\left(\frac{1+\sqrt{5}}{2}\right)\cdot\left(\frac{1+\sqrt{5}}{2}\right)\cdot\left(\frac{1+\sqrt{5}}{2}\right) = \frac{4+2\sqrt{5}}{2}$		$\left(\frac{1-\sqrt{5}}{2}\right)\cdot\left(\frac{1-\sqrt{5}}{2}\right)\cdot\left(\frac{1-\sqrt{5}}{2}\right) = \frac{4-2\sqrt{5}}{2}$
n=4	$\left(\frac{1+\sqrt{5}}{2}\right)^4 = \frac{7+3\sqrt{5}}{2}$		$\left(\frac{1-\sqrt{5}}{2}\right)^4 = \frac{7-3\sqrt{5}}{2}$
n=5	$\left(\frac{1+\sqrt{5}}{2}\right)^5 = \frac{11+5\sqrt{5}}{2}$		$\left(\frac{1-\sqrt{5}}{2}\right)^5 = \frac{11-5\sqrt{5}}{2}$
n=6	$\left(\frac{1+\sqrt{5}}{2}\right)^6 = \frac{18+8\sqrt{5}}{2}$		$\left(\frac{1-\sqrt{5}}{2}\right)^6 = \frac{18-8\sqrt{5}}{2}$

Figure 61 – Binomial Expansion used for the Fibonacci Numbers. Note there are two coefficients for each n , one for the rational part and one for the irrational part.

$t_u \downarrow$ $t_o \downarrow$ $t_u \downarrow$ $t_o \downarrow$ $f_n \downarrow$

n=1	$\frac{1}{\sqrt{5}}\cdot\left(\frac{1+1\sqrt{5}}{2}-\frac{1-1\sqrt{5}}{2}\right)=1$
n=2	$\frac{1}{\sqrt{5}}\cdot\left(\frac{3+1\sqrt{5}}{2}-\frac{3-1\sqrt{5}}{2}\right)=1$
n=3	$\frac{1}{\sqrt{5}}\cdot\left(\frac{4+2\sqrt{5}}{2}-\frac{4-2\sqrt{5}}{2}\right)=2$
n=4	$\frac{1}{\sqrt{5}}\cdot\left(\frac{7+3\sqrt{5}}{2}-\frac{7-3\sqrt{5}}{2}\right)=3$
n=5	$\frac{1}{\sqrt{5}}\cdot\left(\frac{11+5\sqrt{5}}{2}-\frac{11-5\sqrt{5}}{2}\right)=5$
n=6	$\frac{1}{\sqrt{5}}\cdot\left(\frac{18+8\sqrt{5}}{2}-\frac{18-8\sqrt{5}}{2}\right)=8$

Figure 62: Coding of the Fibonacci Numbers. t_u subtracts out to zero but determines the next f_n .

Figure 62 contains the calculation for the first six Fibonacci Numbers. The first thing we note is that the function produces the correct Fibonacci integer f_n .

We also see the rational and irrational parts have two tracking integers t_u and t_o. t_o is observable and provides the value of the Fibonacci Number f_n. t_u is unobservable and subtracts out to zero. From Figure 61, for $n = 3$, $t_o = 2$ and $t_u = 4$, and separating the rational and irrational parts we get the results in Fig 62:

$$\frac{1}{\sqrt{5}}\cdot\left(\frac{2\sqrt{5}}{2}-\frac{-2\sqrt{5}}{2}\right)=2=f_3 \quad \text{and} \quad \frac{1}{\sqrt{5}}\cdot\left(\frac{4}{2}-\frac{4}{2}\right)=0$$

Although t_u subtracts out and is unobservable it is still important. It is needed to determine the next set of values. If we add t_u to t_o and divide by 2 we get the next Fibonacci number. For $n = 3$: $\frac{4+2}{2}=3=f_4$.

In general: $$f_{n+1}=\frac{t_u+t_o}{2} \qquad (6.15)$$

In this way nature stores the information for future cycles in the current cycle through binomial expansion of a mixed rational and irrational number. In (6.4) we required the GTE be true at these boundary conditions. (6.13) is the solution to this equation and uses the difference of the two expansions which results in t_u dropping out. Yet it is necessary for it to be in the solution set in order to solve the DE.

The tracking numbers are added to each cycle. Every time we raise n to the next integer that is equivalent to multiplying the n^{th} base by $(1 \pm \sqrt{5})/2$ one more time. Referring to $n = 3$ and $n = 4$, from Figure 61 we see that the fourth power expansion equals the third power times the first power:

$$\left(\frac{1+1\sqrt{5}}{2}\right)^4 = \left(\frac{1+1\sqrt{5}}{2}\right)^3 \cdot \left(\frac{1+1\sqrt{5}}{2}\right)^1$$

$$= \left(\frac{4+2\sqrt{5}}{2}\right) \cdot \left(\frac{1+1\sqrt{5}}{2}\right)$$

$$= \left(\frac{4+2\sqrt{5}+4\sqrt{5}+10}{4}\right) = \left(\frac{7+3\sqrt{5}}{2}\right) \tag{6.16}$$

And in general

$$\left(\frac{1+1\sqrt{5}}{2}\right)^n = \left(\frac{1+1\sqrt{5}}{2}\right)^{n-1} \cdot \left(\frac{1+1\sqrt{5}}{2}\right)^1 \tag{6.17}$$

This is not a special property for this particular number, it applies to all numbers. What is special about this number is that each tracking integer is the sum of the two before it. So not only does the Fibonacci number follow this rule, the tracking numbers do as well. So in addition to the previous equation:

$$\left(\frac{1+1\sqrt{5}}{2}\right)^n = \left(\frac{1+1\sqrt{5}}{2}\right)^{n-1} + \left(\frac{1+1\sqrt{5}}{2}\right)^{n-2} \tag{6.18}$$

D. Determination of Boundary Conditions

We have thus far used $(1 \pm \sqrt{5})/2$ as the boundary conditions without explanation. The operation in Equation (6.17) could have applied to expanding any number. That of (6.18) is specific to the Fibonacci numbers as it requires the current value to be equal to the sum of the two previous values. $(1 \pm \sqrt{5})/2$ are the special numbers that work in Equation (6.18). If they were unknown we could solve for them. Let y be the special number. Then (6.18) becomes:

$$(y)^n = (y)^{n-1} + (y)^{n-2} \tag{6.19}$$

The equation is true for all n so we can choose any value. If we let $n = 2$ we get $y^2 = y^1 + y^0$ or

$$y^2 - y - 1 = 0 \tag{6.20}$$

The solutions to this equation are of course $y = (1 \pm \sqrt{5}) / 2$. (6.21)

Therefore the boundary conditions A and B are selected as $(1 \pm \sqrt{5}) / 2$ to satisfy $y^n = y^{n-1} + y^{n-2}$ which is the Fibonacci relation, add the previous two values to get the next.

E. The Golden Ratio

The Fibonacci numbers are associated with the Golden Ratio and the infinite rectangle pattern. Here we have an outer rectangle with sides y and x with an infinite number of inset rectangles of the same proportion. The ratio of the sides of outermost rectangle is $y : x$, and for the first inset rectangle $x : (y - x)$. If these two rectangles are of the same proportion then $y / x = x / (y - x)$.

This is true for all x. If we let $x = 1$ this reduces to $y = 1 / (y - 1)$ which equals $y^2 - y - 1 = 0$. This is the same as Equation (6.20) with solution $y = (1 \pm \sqrt{5}) / 2$. Only the positive solution is applicable in this case. So the golden ration is:

$$(1 + \sqrt{5}) / 2 \qquad (6.22)$$

The presence of the irrational component $\sqrt{5}$ indicates the problem is more complex than real space allows us to observe. When natural phenomena follow Fibonacci Numbers or the Golden Ratio they are following the rules set forth by the Equivalence Relation or GTE. The solution set contains an unobservable tracking number used to determine the next value. In this manner the entire series is encoded in a single equation. The key to the operation is the boundary conditions in the Equivalence Relation. These are determined by requiring that the value at any cycle be equal to the sum of the previous two cycles, which is simply the definition of the Fibonacci Series. Thus the GTE derives the Fibonacci Equation from first principles.

It also provides a means to implement this. The GTE sets requirements for a specific action analogous to the equations that determine the allowable modes of a guitar string. When the action occurs it must conform to the rules of a differential equation and there are specific allowable states. Here there is a relationship of addition between cycles and this dictates what states, or Fibonacci Numbers, the differential equation will allow.