Part IV. Generation of Some π Squared Series

Here we generate other well known series for π squared:

$$\frac{\pi^2}{8}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\ ,\quad \frac{\pi^2}{12}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\ ,\quad \text{and}\quad \frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2} \tag{IV.A}$$

Using the same procedure, we seek the function $F(\Theta)$ that solves the GTE for π^2.

$$\int_A^B\int f(\theta)\cdot d\theta\cdot d\theta=\int_A^B\int \mathrm{F}(\Theta)\cdot d\Theta\cdot d\Theta \tag{IV.B}$$

In the previous analysis we used $f(\theta)=1$ with $F(\Theta)=\sum_{n=0}^{\infty}-2\cdot e^{in\cdot(-\Theta+\pi)}$ and $\theta_m=\pi.$

Here we will use $f(\theta)=1$ with $F(\Theta)=\sum_{n=0}^{\infty}4\cdot e^{in\cdot(-\Theta+\pi)}\cdot\left(i^n\right)$ and $\theta_m=\pi.$ (IV.C)

Figure 33 demonstrates graphical equivalence and Figure 34 shows how the terms work. For the left hand side of the GTE:

$$\int_A^B\int f(\theta)d\theta=\int_{-\theta_m/2}^{\theta_m/2}\int d\theta\cdot d\theta=\theta^2\Big|_{-\theta_m/2}^{+\theta_m/2}=\theta_m{}^2 \tag{IV.D}$$

For the right hand side of the GTE:

$$\int_{-\theta_m/2}^{\theta_m/2}\int\sum_{n=0}^{\infty}4\cdot e^{in\cdot(-\Theta+\pi)}\cdot\left(i^n\right)d\Theta^2 \tag{IV.E}$$

$$\sum_{n=0}^{\infty}\frac{-4\cdot e^{in\cdot(-\Theta+\pi)}}{n^2}\cdot\left(i^n\right)\Big|_{-\theta_m/2}^{+\theta_m/2}=\quad -4\cdot\sum_{n=1}^{\infty}\frac{e^{in\cdot(-\theta_m/2+\pi)}-e^{in\cdot(+\theta_m/2+\pi)}}{n^2}\cdot\left(i^n\right)$$

The even terms will cancel and we can begin summation at $n=1$. Substituting (IV.D)&(IV.E) into (IV.B) and $\theta_m=\pi$:

$$\pi^2=-4\cdot\sum_{n=1}^{\infty}\frac{e^{in\cdot(-\pi/2+\pi)}-e^{in\cdot(+\pi/2+\pi)}}{n^2}\cdot\left(i^n\right) \tag{IV.F}$$

Rearranging

$$\frac{\pi^2}{4}=-\sum_{n=1}^{\infty}\frac{e^{in\cdot(\pi-\pi/2)}-e^{in\cdot(\pi+\pi/2)}}{n^2}\cdot\left(i^n\right) \tag{IV.G}$$

Expanding the series and eliminating the zero terms:

$$\frac{\pi^2}{4} = -\left[\left((+i)-(-i)\right)\cdot(i) + \left(\frac{(-1)-(-1)}{4}\right)\cdot(-1) + \left(\frac{(-\mathrm{i})-(+\mathrm{i})}{9}\right)\cdot(-i) + \left(\frac{(+1)-(+1)}{16}\right)\cdot(-i) + \ldots\right]$$

$$= +2\left[1+0+\frac{1}{9}+0+\frac{1}{25}+0+\frac{1}{49}+\ldots\right] \qquad \text{(IV.H)}$$

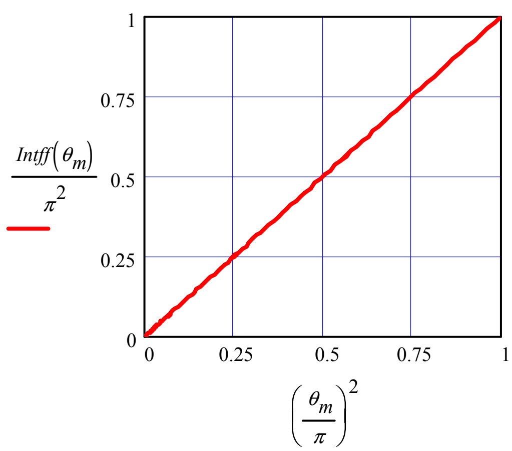
And we find $$\frac{\pi^2}{8} = 1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\ldots = \sum_{n=0}^{\infty}\frac{1}{(2n-1)^2}. \qquad \text{(IV.i)}$$

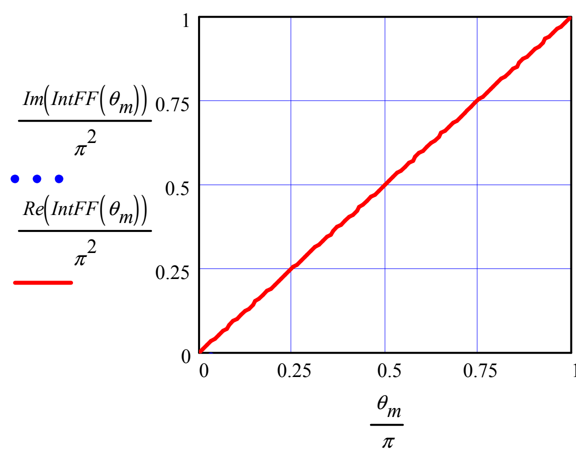
This is the first of the relations we set out to derive. We see that all of the even terms drop out and the odd terms have the factor $\frac{1}{(2n-1)^2}$.

Figure 33 – Graphical Equivalence for the π Squared Function (IV.i). Graphs of the left and right hand side of the GTE are shown. θ_m is the measured angle. When $\theta_m = \pi$, $A = -\pi/2$ and $B = +\pi/2$.

$$Intff(\theta_m) := \theta_m^{\,2}$$

$$IntFF(\theta_m) := \sum_{n=1}^{1000}\left[\left[\frac{-4\cdot\left[e^{\mathrm{i}\cdot n\cdot\left(\pi-\frac{\theta_m}{2}\right)}\right]}{n^2} - \frac{-4\cdot\left[e^{\mathrm{i}\cdot n\cdot\left(\pi+\frac{\theta_m}{2}\right)}\right]}{n^2}\right]\cdot\left(\mathrm{i}^{n}\right)\right]$$





The right side is plotted against $\frac{\Theta}{\pi}$ rather than $\frac{\Theta^2}{\pi^2}$ because Θ appears in the equation as a first order variable.

Figure 34 – Calculations for the π squared Equation

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n-1)^2}$$

1. The integral on the right hand side of the GTE is:

$$IntFF_n := \left[\frac{-4\cdot\left[e^{i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)}\right]}{n^2} - \frac{-4\cdot\left[e^{i\cdot n\cdot\left(\pi+\frac{\pi}{2}\right)}\right]}{n^2}\right]\cdot\left(i^n\right)$$

2. The value of this integral is:

$$\sum_{n=1}^{1000}\left[\left[\frac{-4\cdot\left[e^{i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)}\right]}{n^2} - \frac{-4\cdot\left[e^{i\cdot n\cdot\left(\pi+\frac{\pi}{2}\right)}\right]}{n^2}\right]\cdot\left(i^n\right)\right] = 9.866 \quad = \quad \pi^2$$

3. The equation also provides the combination of harmonic components for each n:

$\frac{IntFF_n}{8} =$

1
0
0.111
0
0.04
0
0.02
0
0.012

4. This combination can be written as a series:

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...$$

5. This can be expressed in summation form:

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n-1)^2}$$

6. The integral on the left hand side of the GTE = -1/2 π^2 was factored out.

Note how the term $\left(i^n\right)$ works to adjust the sign of the non-zero factors:

A - B = C x D = E

A: $\frac{\frac{-4\cdot\left[e^{i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)}\right]}{n^2}}{8} =$	B: $\frac{\frac{-4\cdot\left[e^{i\cdot n\cdot\left(\pi+\frac{\pi}{2}\right)}\right]}{n^2}}{8} =$	C: $\frac{\left[\frac{-4\cdot\left[e^{i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)}\right]}{n^2} - \frac{-4\cdot\left[e^{i\cdot n\cdot\left(\pi+\frac{\pi}{2}\right)}\right]}{n^2}\right]}{8}$	D: $i^n =$	E: $\frac{IntFF_n}{8} =$
-0.5i	0.5i	-i	i	1
0.125	0.125	0	-1	0
0.056i	-0.056i	0.111i	-i	0.111
-0.031	-0.031	0	1	0
-0.02i	0.02i	-0.04i	i	0.04
0.014	0.014	0	-1	0
0.01i	-0.01i	0.02i	-i	0.02
$-7.813\cdot10^{-3}$	$-7.813\cdot10^{-3}$	0	1	0
$-6.173i\cdot10^{-3}$	$6.173i\cdot10^{-3}$	-0.012i	i	0.012

For the second series we replace n^2 with $(n+1)^2$ in the denominator of (IV.F) and normalize. Note the sign changing term $\left(i^n\right)$ is not needed here:

$$\pi^2 = -24i \cdot \sum_{n=1}^{\infty} \frac{e^{in\cdot(-\pi/2+\pi)} - e^{in\cdot(+\pi/2+\pi)}}{(n+1)^2} \tag{IV.J}$$

Expanding the series and eliminating the zero terms:

$$\frac{\pi^2}{24} = -i \cdot \left[\left(\frac{(+i)-(-i)}{2^2}\right) + \left(\frac{(-1)-(+1)}{3^2}\right) + \left(\frac{(-i)-(+i)}{4^2}\right) + \left(\frac{(+1)-(-1)}{5^2}\right)....\right]$$

$$= -i \cdot \left[\left(\frac{2i}{4}\right) + 0 + \left(\frac{-2i}{16}\right) + 0 + \left(\frac{2i}{36}\right)....\right] \qquad = 2\left[1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16}....\right] \tag{IV.K}$$

And we find our second series
$$\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + ... = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n^2}. \tag{IV.L}$$

In Figure 35 we demonstrate graphical equivalence and Figure 36 contains the calculation.

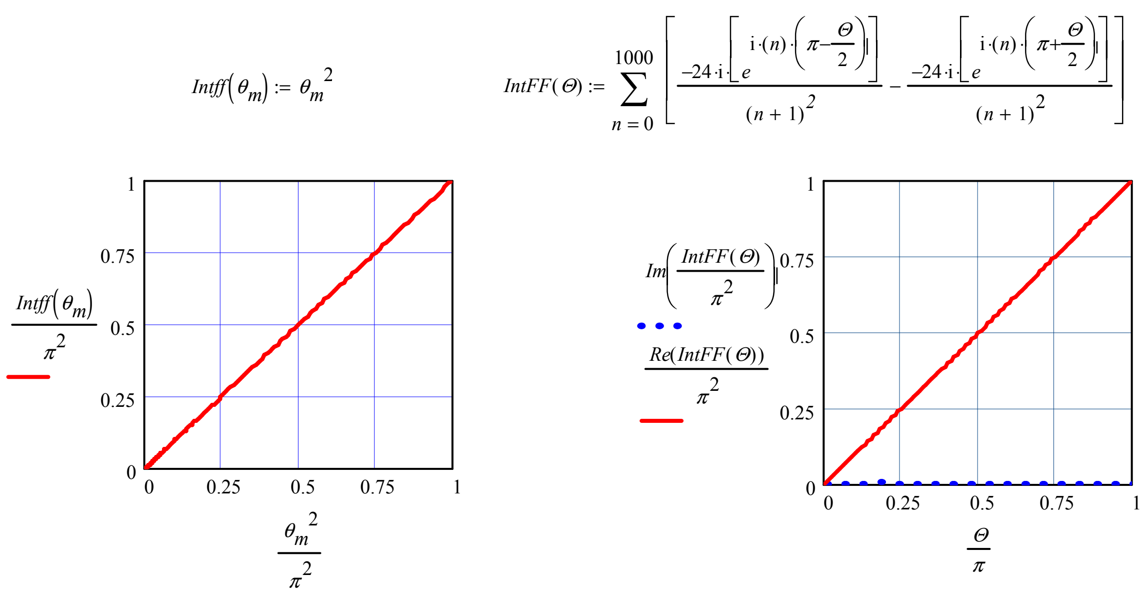


Figure 35 – Graphical Equivalence for (IV.J)

Figure 36 – Calculations for the π^2 Equation (IV.J)

1. The integral on the right hand side of the GTE is:

$$IntFF_n := \frac{-24\cdot i\cdot\left[e^{i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)}\right]}{(n+1)^2} - \frac{-24\cdot i\cdot\left[e^{i\cdot n\cdot\left(\pi+\frac{\pi}{2}\right)}\right]}{(n+1)^2}$$

2. The value of this integral is:

$$\sum_{n=1}^{1000}\left[\frac{-24\cdot i\cdot\left[e^{i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)}\right]}{(n+1)^2} - \frac{-24\cdot i\cdot\left[e^{i\cdot n\cdot\left(\pi+\frac{\pi}{2}\right)}\right]}{(n+1)^2}\right] = 9.87 \qquad = \pi^2$$

3. The equation also provides the combination of harmonic components for each n:

$\dfrac{\frac{-24\cdot i\cdot\left[e^{i\cdot n\cdot\left(\pi-\frac{\pi}{2}\right)}\right]}{(n+1)^2}}{12}$	$\dfrac{\frac{-24\cdot i\cdot\left[e^{i\cdot n\cdot\left(\pi+\frac{\pi}{2}\right)}\right]}{(n+1)^2}}{12}$	$\dfrac{IntFF_n}{12} =$
0.5	-0.5	1
0.222i	0.222i	0
-0.125	0.125	-0.25
-0.08i	-0.08i	0
0.056	-0.056	0.111
0.041i	0.041i	0
-0.031	0.031	-0.063
-0.025i	-0.025i	0
0.02	-0.02	0.04
0.017i	0.017i	0

4. This combination can be written as a series:

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

5. This can be expressed in summation form:

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

6. The integral on the left hand side of the GTE = π^2 was factored out.

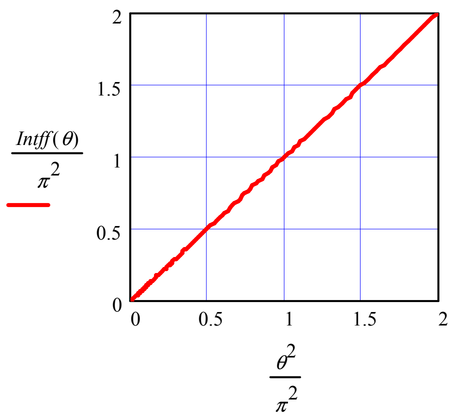
The third series is the famous Basel Problem. We note that we can obtain this series by sign changing every $4n-3$ term in the final table in Figure 36. We can do this by introducing the sign changing term $\left(\frac{i^{n-1}-i^{-n-1}}{2}\right)$. This is more complex looking than a $\left(i^{n-1}\right)$ formula but it zeros at the even n. Substituting into (IV.J) and renormalizing:

$\left(\frac{i^{n-1}-i^{-n-1}}{2}\right) =$
1
0
-1
0
1
0
-1
0
1
0

$$\pi^2 = -12i\cdot\sum_{n=1}^{\infty}\frac{e^{in\cdot\left(-\pi/2+\pi\right)} - e^{in\cdot\left(+\pi/2+\pi\right)}}{(n+1)^2}\cdot\left(\frac{i^{n-1}-i^{-n-1}}{2}\right) \qquad \text{(IV.M)}$$

In Figure 37 (IV.M) is graphed and compared to the real function. As can be seen (IV.M) ***is not*** graphically equivalent over the range. However it is graphically equivalent at $\Theta = \pi$. We can use this provided we restrict ourselves to this value only.

$$Intff(\theta) := \theta^2 \qquad IntFF(\Theta) := \sum_{n=0}^{1000} \left[\left[\frac{-12 \cdot i \cdot \left[e^{i \cdot (n) \cdot \left(\pi - \frac{\Theta}{2} \right)} \right]}{(n+1)^2} - \frac{-12 \cdot i \cdot \left[e^{i \cdot (n) \cdot \left(\pi + \frac{\Theta}{2} \right)} \right]}{(n+1)^2} \right] \cdot \left(\frac{i^{n-1} - i^{-n-1}}{2} \right) \right]$$



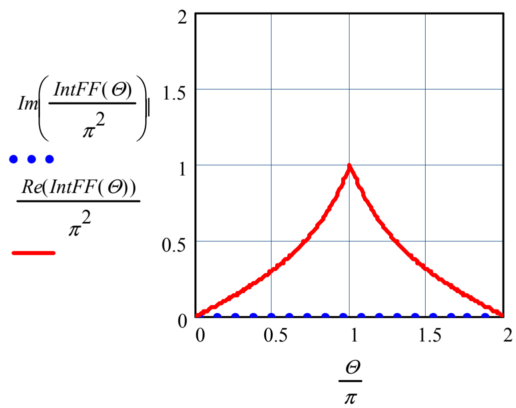


Figure 37 – Equation (IV.M) is not graphically equivalent over the range but it is graphically equivalent at the point $\Theta = \pi$.

Figure 38 – Calculations for the π^2 Equation (IV.P)

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

1. The integral on the right hand side of the GTE is:

$$IntFF_n := \left[\left[\frac{-12 \cdot i \cdot \left[e^{i \cdot (n) \cdot \left(\pi - \frac{\pi}{2} \right)} \right]}{(n+1)^2} - \frac{-12 \cdot i \cdot \left[e^{i \cdot (n) \cdot \left(\pi + \frac{\pi}{2} \right)} \right]}{(n+1)^2} \right] \cdot \left(\frac{i^{n-1} - i^{-n-1}}{2} \right) \right]$$

2. The value of this integral is:

$$\sum_{n=1}^{1000} \left[\left[\frac{-12 \cdot i \cdot \left[e^{i \cdot n \cdot \left(\pi - \frac{\pi}{2} \right)} \right]}{(n+1)^2} - \frac{-12 \cdot i \cdot \left[e^{i \cdot n \cdot \left(\pi + \frac{\pi}{2} \right)} \right]}{(n+1)^2} \right] \cdot \left(\frac{i^{n-1} - i^{-n-1}}{2} \right) \right] = 9.858 \qquad = \pi^2$$

3. The equation also provides the combination of harmonic components for each *n*:

$\frac{-12 \cdot i \cdot \left[e^{i \cdot (n) \cdot \left(\pi - \frac{\pi}{2} \right)} \right]}{(n+1)^2}$	$\frac{-12 \cdot i \cdot \left[e^{i \cdot (n) \cdot \left(\pi + \frac{\pi}{2} \right)} \right]}{(n+1)^2}$	$\left[\frac{-12 \cdot i \cdot \left[e^{i \cdot (n) \cdot \left(\pi - \frac{\pi}{2} \right)} \right]}{(n+1)^2} - \frac{-12 \cdot i \cdot \left[e^{i \cdot (n) \cdot \left(\pi + \frac{\pi}{2} \right)} \right]}{(n+1)^2} \right]$
3	-3	6
1.333i	1.333i	0
-0.75	0.75	-1.5
-0.48i	-0.48i	0
0.333	-0.333	0.667
0.245i	0.245i	0
-0.188	0.188	-0.375
-0.148i	-0.148i	0
0.12	-0.12	0.24

$\frac{i^{n-1} - i^{-n-1}}{2}$	$\frac{IntFF_n}{6} =$
	1
1	0
0	0.25
-1	0
0	0.111
1	0
0	0.063
-1	0
0	0.04
1	

4. This combination can be written as a series:

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...$$

5. This can be expressed in summation form:

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

6. The integral on the left hand side of the GTE = π^2 was factored out.

Figure 38 shows the calculation and normalization. Expanding the series (IV.M) we obtain:

$$\frac{\pi^2}{12} = -i\cdot\left[\left(\frac{(+i)-(-i)}{2^2}\right)\cdot(+1)+\left(\frac{(-1)-(+1)}{3^2}\right)\cdot 0+\left(\frac{(-i)-(+i)}{4^2}\right)\cdot(-1)+\left(\frac{(+1)-(-1)}{5^2}\right)\cdot 0....\right]$$

$$= -2i\cdot\left[\left(\frac{2i}{4}\right)+0+\left(\frac{2i}{16}\right)+0+\left(\frac{2i}{36}\right)....\right] \qquad = 2\left[1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}....\right] \qquad \text{(IV.N)}$$

And we find our third series $$\frac{\pi^2}{6} = 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+.... = \sum_{n=1}^{\infty}\frac{1}{n^2}. \qquad \text{(IV.O)}$$

Since the sign change resulted in $\pi^2/12$ going to $\pi^2/6$ which is a difference of $\pi^2/12$, the value of the terms that were initially subtracted out and then added in is $\pi^2/24,$ so

$$\frac{\pi^2}{24} = \frac{1}{4}+\frac{1}{16}+\frac{1}{36}.... = \sum_{n=1}^{\infty}\frac{1}{(2n)^2} \qquad \text{(IV.P)}$$

Since graphical equivalence was not achieved with (IV.M), this was not an equivalence condition. The study is incomplete and will be revisited. However with partial equivalence we were still able to generate the series which was our objective in this section.