Part 1: *e*

A. Definition of e^x

We will use the function $f(x) = e^x$ which we define to be the function that is equal to its own derivative. That is to say that if we were to pick any point x on the curve we would find that the corresponding value $f(x)$ is also equal to the rate that $f(x)$ is increasing, or equivalently, the slope of a straight line drawn tangent to the curve at this point. This pattern often occurs in nature. We might expect the growth in insect population to be proportional to the number of insects we currently have, or the dying off or decline of a natural phenomena to also be proportional to its current value.

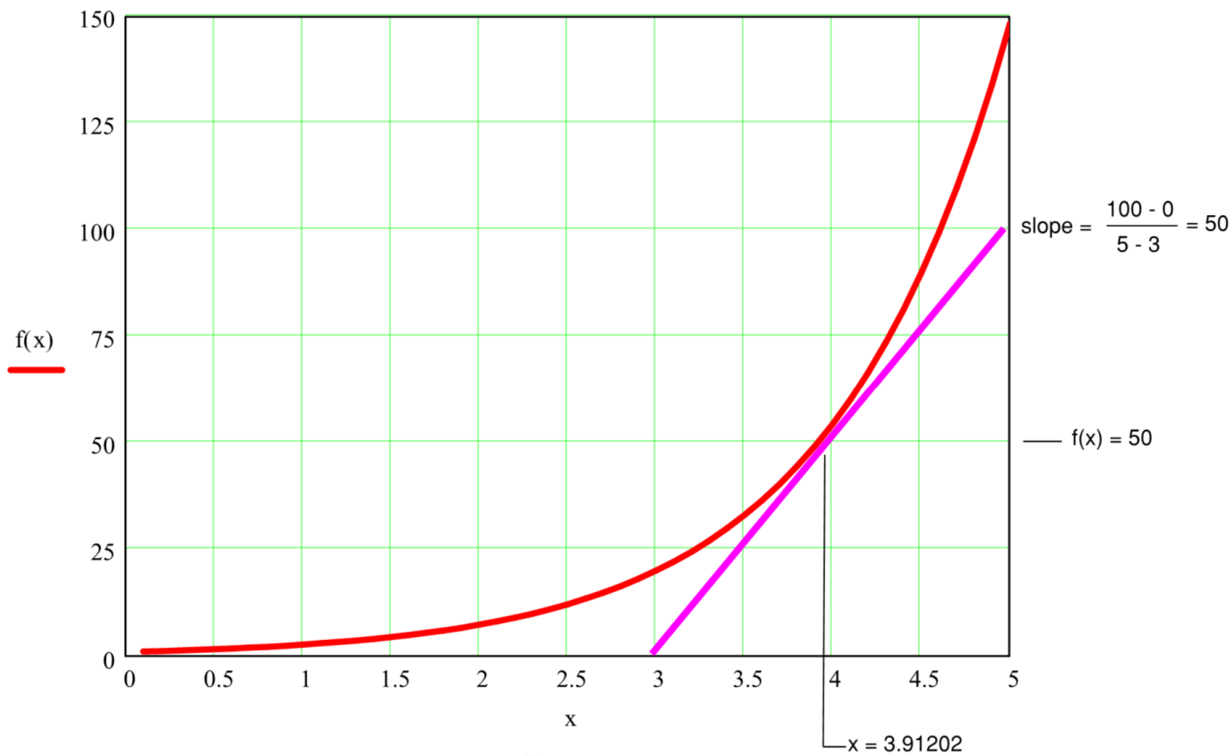


Figure 3

Graphically we depict this in Figure 3. If we draw a line tangent to the curve at any point, the slope of the tangent line represents the rate that curve is increasing. At $x = 3.91202\ldots$, $f(x) = 50$ so the value of the function is 50 at this point. The slope (rise divided by run) of the tangent line also has a value of 50 at this point, so at $x = 3.91202\ldots$, $f'(x) = 50$ as well. This relationship holds for all values of x.

A better mathematical description for the derivative would involve evaluating $f(x)$ at a point slightly less than x and a point slightly greater than x, each differing from x by an amount Δx, and subtracting these values to get the rise. The run would be $2\Delta x$, and so the derivative would be

$$f'(x) = \frac{e^{x+\Delta x} - e^{x-\Delta x}}{2\Delta x}.$$

As noted, the value of $f'(x)$ is equal to $f(x)$ for all x. We can express this as $f'(x) = f(x)$ where $f(x)$ is our function e^x and $f'(x)$ is the derivative. This is our definition of e^x. It is the function that is its own derivative. This differs from the traditional definition of e^x but is mathematically equivalent.

B. Importance & Motivation

Why is the function $f(x) = e^x$ being equal to its rate of change $f'(x)$ so important? First, it is because such relationships occur frequently in nature. For example, the growth rate of an unchecked animal population at any time might be equal (or proportional) to the population. Or, in describing decay, the rate a radioactive material decays would be equal (or proportional) to the amount of material present. Since this function e^x is so important we are naturally inclined to ask what the underlying value of e is. And, therein lies the problem. e is in fact an irrational number, $e = 2.718 \ldots$. How on earth can this number which is integral to natural phenomena be so unnatural that it is irrational? Even more perplexing, it can be expressed as the sum of infinite components:

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \ldots \qquad \text{or in summation notation } e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

This is the formula we will derive. The messy irrational number e in real space is a natural outcome of the function being equal to its rate of change when it is mapped in harmonic space. The relation allows a physical event to be described by an alternate set of variables which map the event for another observer in the alternate space. The two different observers must agree on the outcome when they are viewing the same event and in this way the two spaces are related. The harmonic space, however, is rich in complexity and has rules of combination that must be obeyed when describing a physical event in the same way we might have rules describing falling bodies in real space. These rules impact the way the process occurs and are integral to it. Surprisingly the rules in harmonic space can have a physical manifestation in real space, even though we cannot observe the spatial variables themselves. This is because both are views of the same event, and the harmonic space has rules that the process must obey. So although we cannot view harmonic space from real space, there are still signs of it. Most noteworthy is that MHA determines the value of the fundamental constants e and π.

x is the variable we will use in real space and $f(x)$ is the mapping function in real space. Thus $f(x) = e^x$. Harmonic space is based on the variable X raised to the n^{th} power and the n^{th} harmonic has a base of X^n. The n individual harmonics are summed to form $F(x)$. This is the mapping function in harmonic space. A physical event measured in either system must yield the same result.

X is in fact unit-less and has a value of 1 when we attempt to view it in real space. This means it is unobservable in real space since $1^n = 1$ for all n. The value of e cannot be determined by an equation in real space since there are no variables in real space. For that reason the harmonic space cannot be observed.

C. Background & Development

$f(x) = e^x$ is the function in real space (x space) that equals its own derivative:

$$f(x) = e^x \quad \text{and} \quad f'(x) = e^x, \quad \text{therefore } f'(x) = f(x)$$

That is our definition of e^x.

We have an event that occurs in real space from point A to point B. We must find a function *F(X)* in harmonic space (X space) that will create an equivalent mapping for that event. That is $F(X)$ must fulfill the requirement:

GTE
$$\int_A^B f(x)dx = \int_A^B F(X)dX \qquad (A)$$

Note that the left side of the equation occurs in x space while the right side occurs in X space. We use A and B to represent the start and end points of the event, also called the boundary conditions. The equation requires that results obtained in x space must equal that in X space when tallied for the action $A \to B$. This is the General Transformation Equation (GTE).

Since the integrals are performed in their respective spaces x and X , it may appear that we are adding apples and getting oranges. That is not the case. The integrals in Equation (A) represent the conclusion reached after using either apples or oranges for measurement. As an analogy, rival fans experience and measure a game in vastly different ways, but at the end of the day when each sums their own measurement and experience, both arrive at the same final score. That is what the equation is saying.

f(x) we know equals e^x. This is the left hand side of Equation (A) which occurs in real space. We need to find the function *F(X)* in harmonic space to put into the right side of Equation (A) so that the results of a physical event measured in the two spaces space agree.

To find the *F(X)* that solves Equation (A) we do not need to execute the integrals of that equation in this particular case. We are working with e^x which is the function that is its own derivative. Since integration and differentiation are inverse operations, not only is *f(x)* its own derivative, it is also its own integral:

$$f'(x) = f(x) \quad \text{implies} \quad \int f(x)dx = f(x) \qquad \text{for } f(x) = e^x \text{ only}$$

Since the integral is equal to the underlying function in this particular case, in the GTE we can replace the integral with its underlying function and avoid integration. So in this specific case Equation (A) can be simplified to require that the two underlying functions *f(x)* and *F(X)* must be equal. Therefore the requirement to satisfy Equation (A) can be simplified:

General Transform Equation
$$\int_A^B f(x)dx = \int_A^B F(X)dX \qquad (A)$$

for $f(x) = e^x$ only, implies
$$f(x) =_g F(X) \qquad (B)$$

Equation (B) should be understood to mean that our function *F(X)* mapped in harmonic space must be graphically equivalent to *f(x)* mapped in real space. That means when *F(X)* is mapped in X space the graph obtained is identical to that when *f(x)* is mapped in x space. In that sense they are equal, but that is the end of the similarity. *f(x)* describes a natural phenomena in real space where growth or decay rate is proportional to the quantity present. *F(X)* is an infinite series of waves that add up to give the same result. Clearly Equation (B) is not an equation we could use algebraic operations on, moving terms across the equals sign with algebraic operations. Since Equation (B) is only about equivalence in mapping values we use the sign "$=_g$" to mean graphically equivalent.

Equation (B) also implies that *F(X)* is its own derivative in X space. This must be true since *f(x)* is its own derivative in x space and the two functions are graphically equivalent when mapped in their own spaces. This implies Equation (C):

Implied by (B) $$F'(X) = F(X) \qquad \text{(C)}$$

This is a differential equation in X space that can be solved. We are looking for the function *F(X)* which is its own derivative. We will try as a solution to Equation (C):

$$F(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n \qquad \text{(D)}$$

F(X) is an infinite series with each component wave function given a number n. Each wave function is the product of an amplitude factor and a harmonic base of X^n. The amplitude factor for each individual component is $\frac{1}{n!}$.

To demonstrate that (D) is the solution to the differential equation (C) we expand the series of harmonics described by (D) to show the components:

$$F(X) = 1 + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \frac{1}{24}X^4 + \frac{1}{120}X^5 + \ldots \qquad \text{(E)}$$

Taking the derivative of each component in the series:

$$F'(X) = 0 + 1 + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \frac{1}{24}X^4 + \frac{1}{120}X^5 + \ldots \qquad \text{(F)}$$

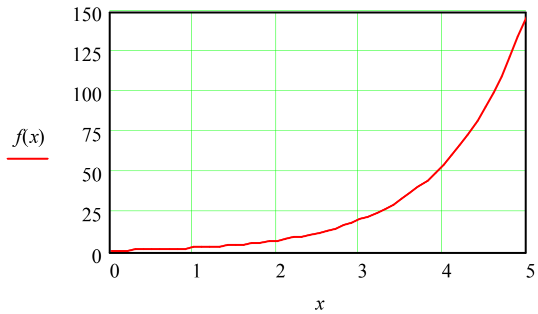
We notice that *F'(X)* does in fact equal *F(X):*

$$1 + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \frac{1}{24}X^4 + \frac{1}{120}X^5 + \ldots \;=\; 0 + 1 + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \frac{1}{24}X^4 + \frac{1}{120}X^5 + \ldots \qquad \text{(G)}$$

Therefore (D) is in fact the solution to the differential equation (C). If we graph the functions *f(x)* and *F(X)* we can see the graphs are identical (See Figure 4).

$$f(x) := e^x \qquad F(X) := \sum_{n=0}^{100} \frac{X^n}{n!}$$

$$f(3) = 20.086 \qquad F(3) = 20.086$$



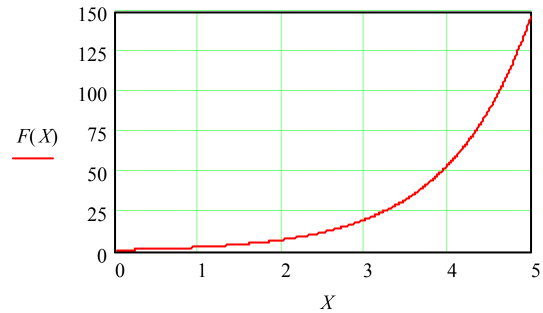


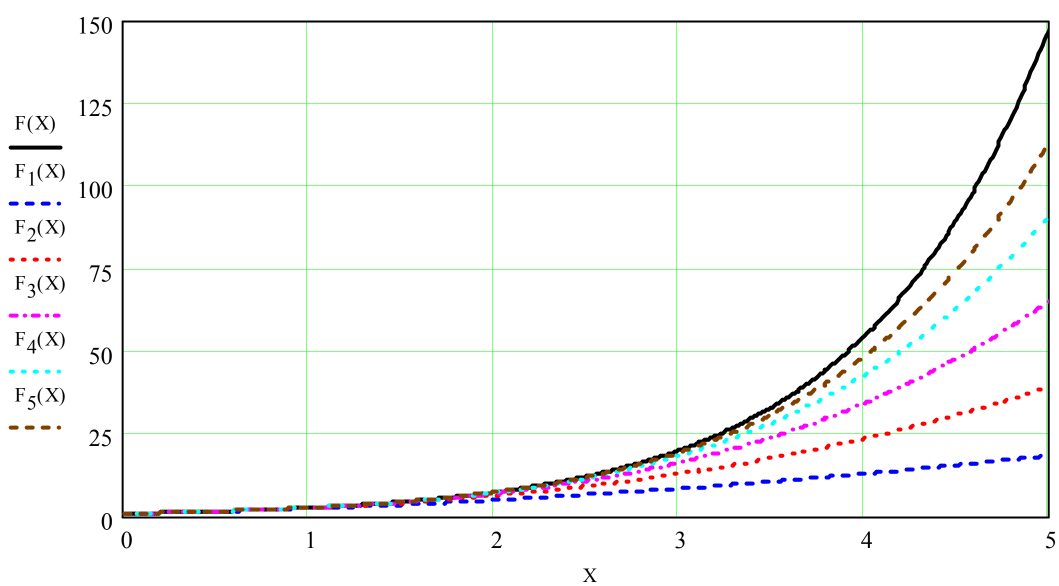
Figure 4 – *f(x)* and *F(X)* are graphically equivalent. This sum of harmonics maps identical to e^x.

This is surprising. What is perhaps the simplest natural function, e^x, can also be defined as an infinite sum of harmonics with each harmonic having an amplitude of $\frac{1}{n!}$.

Figure 5 shows progressive summing of modes 2 though 6 and *F(X)* which is the sum of all of the modes and is equal to *f(x)*. Each additional mode adds a smaller amount resulting in convergence.

<u>Figure 5</u> – Summation of Modes 2 – 6 illustrates convergance.

$$F_1(X) := \sum_{n=0}^{2} \frac{X^n}{n!} \quad F_2(X) := \sum_{n=0}^{3} \frac{X^n}{n!} \quad F_3(X) := \sum_{n=0}^{4} \frac{X^n}{n!} \quad F_4(X) := \sum_{n=0}^{5} \frac{X^n}{n!} \quad F_5(X) := \sum_{n=0}^{6} \frac{X^n}{n!}$$



We call this modal harmonics since each of the n modes that X undertakes (X^0, X^1, X^2, X^3... X^n) is of greater order and smaller amplitude $(1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120})$. See Equation (E) and Figure 6.

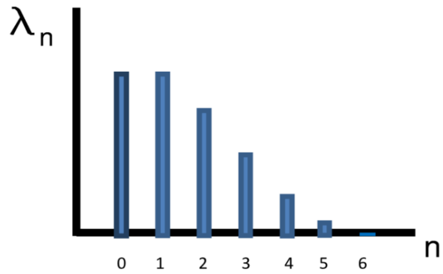


Figure 6

As an analogy we look to Fourier's transformation of a signal in time as being the sum of progressively finer wave modes each with its own amplitude coefficient.

To summarize *F'(X) = F(X)* is the differential equation (C) with solution $F(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n$ (D) that the GTE requires for physical actions involving *e*. This solution can be seen to be the sum of wave forms, each made up of two factors two factors: $\frac{1}{n!}$ and X^n. The harmonic base is given by X^n and gives the wave shape. The factor $\frac{1}{n!}$ is the amplitude factor which determines how large a contribution that particular harmonic makes. We rewrite the amplitude factor as λ_n where $\lambda_n = \frac{1}{n!}$.

The solution to the Equation (C) DE can therefore be expressed as:

$$F(X) = \sum_{n=0}^{\infty} \lambda_n X^n \quad \text{with} \quad \lambda_n = \frac{1}{n!} \quad \text{(amplitude factor)} \qquad \text{(H)}$$

This infinite sum of this harmonics series produces a function *F(X)* in harmonic space that is graphically equal to its counterpart *f(x)* in real space. However the individual harmonics must combine in a specific way for this to happen. The requirement is that

$$\lambda_n = \frac{1}{n!} \qquad \text{(I)}$$

We observe from Equations (E), (F) and (G) that an amplitude factor of $\frac{1}{n!}$ applied to the nth harmonic is the key to having the derivative equal to the function in harmonic space, or *F'(X)=F(X).*

λ_n must have this value. If it did not then when we sum the $\lambda_n X^n$ we would not get a function that is equal to its own derivative. Then *F(X)* would not be graphically equivalent to *f(x)*. This would violate the GTE (A) and is forbidden.

D. Modal Harmonic Equations for Determination of the Fundamental Constant *e*

Let $f(x) = e^x$. This is the equation in real space we seek the transformation of. *f(x)* is defined as the function that equals its own derivative, thus $f'(x) = f(x)$. Let *F(X)* be the transformed function in harmonic space that we seek. According to the GTE it is required that when we measure a physical phenomenon we get the same result regardless of which space is used:

General Transform Equation $$\int_A^B f(x)dx = \int_A^B F(X)dX \qquad (1)$$

This is the General Transformation Equation (GTE). However, in this special case where $f(x) = e^x$, the derivative of the function has the same value as the function, or $f'(x) = f(x)$. This implies that the integral of *f(x)* also have the same value of *f(x)* since differentiation and integration are inverse operations. That is the left hand side of Equation (1) operating in x space. In order for Equation (1) to be true for the action between any two points A and B common to both spaces, *F(X)* must be *graphically equivalent* to *f(x)* and we can relate the underlying functions of the GTE without evaluating the GTE integrals. So for $f(x) = e^x$ only Equation (1) implies:

$$f(x) =_g F(X) \qquad (2)$$

The symbol $=_g$ means graphically equivalent. By this we mean they have the same numerical result when given the same input. We cannot, however, do algebraic operations across the equals sign because the left and right sides are of different spaces.

F(X) must also be its own derivative since *f(x)* is its own derivative and they are graphically the same:

$$F'(X) = F(X) \qquad (3)$$

Equation (3) is a differential equation (DE) in harmonic space. The solution is:

$$F(X) = \sum_{n=0}^{\infty} \lambda_n X^n \quad \text{with} \quad \lambda_n = \frac{1}{n!} \quad \text{(amplitude factor)} \qquad (4)$$

This represents the sum of n harmonic modes (X^0, X^1, X^2, etc.) with an amplitude factor λ_n for each. When summed this function in X space graphs identically to *f(x)* in x space even though it is composed of individual wave functions. So long as we choose the same input value for *x* and *X* we will get the same results from *f(x)* and *F(X)*.

Returning to Equation (2): $f(x) =_g F(X)$, we substitute using $f(x) = e^x$ and $F(X) = \sum_{n=0}^{\infty} \lambda_n X^n$:

$$e^x =_g \sum_{n=0}^{\infty} \lambda_n X^n \qquad (5)$$

We can choose any value of x and X for evaluation so long as we give both x and X the same value.

Since we are looking for the value of *e* and not a power of it, let $x = 1$ and $X = 1$ in Equation (5). Then $X^n = 1$ for all n and the harmonic component will drop out:

$$e^1 = \sum_{n=0}^{\infty} \lambda_n \cdot 1^n \quad \Rightarrow \quad e = \sum_{n=0}^{\infty} \lambda_n \tag{6}$$

e <u>must</u> have this value since Equation (5) was a substitution into Equation (2), which is a form of the GTE for this particular case. Thus MHA requires that the value of *e* in real space be the sum of the eigenvalues that form the solution to the Equation (3) DE in harmonic space.

Substituting $\lambda_n = \frac{1}{n!}$ from (4) into (6):

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \tag{7}$$

<u>This is the formula we set out to derive.</u> *e* <u>must</u> have this value or the GTE would be violated.

We have returned to the standard equals symbol in (6) and (7) because the harmonic has dropped out and algebraic operations are permitted. Note that by selecting $x = 1$ and $X = 1$ for evaluating (5) the harmonic space disappears from the solution. This is not trivial. Solving the equation requires that harmonic space be unobservable from real space. Thus in the transition from (5) to (6) we go from graphically equivalent to equal. But the solution also requires that traces of the harmonic space be directly observable, namely in setting the value of *e*. *e* must equal the sum of the modal amplitudes λ_n which are the eigenvalues that solve the differential equation $F'(X) = F(X)$ in harmonic space.

Since either space could be used to map the same event, physics requires that they both produce the same result when evaluated over the same action. This is the meaning of the GTE. Harmonic space is richer and more complex than real space and only a specific solution set will cause the harmonics to add up to create the function that is its own derivative and solve the DE (3) which the GTE requires. ***Therefore the harmonic space requires that the physical phenomena occur in a certain way and there are specific rules of combination. Additionally the mechanics behind the action cannot be viewed in real space, similar to what we saw in the Spring Theatre example.***

At deeper level we could say that *e* must have a value assigned to it for natural phenomena to operate, but there are insufficient variables in real space to do this. Tasked with this problem nature relies on a virtual space to control and access this value. The counterpart of e^x in harmonic space is a DE which is solved only when the harmonics add up in an exact way. This can be likened to a lock with *n* pins with the key contained in the solution to the DE. The requirement is that the n^{th} harmonic must have an amplitude factor of 1/*n*! :

$$\lambda_n = \frac{1}{n!} \tag{8}$$

This is the method that nature uses for storing and accessing the fundamental constant *e*. Any time that a phenomena occurs that utilizes *e* it invokes this DE in the same way that plucking a guitar string would invoke that DE. The solution to the DE contains the coding pattern as the amplitude factors for the individual harmonics, the DE is the method of storing the pattern in a timeless and indestructible way, and the GTE is the method of accessing and executing it.