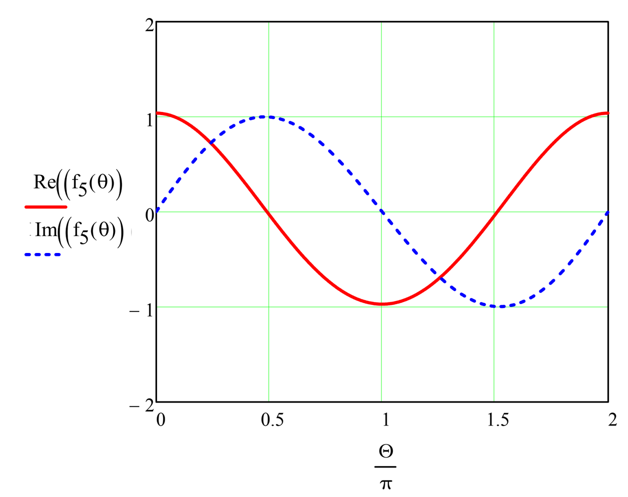
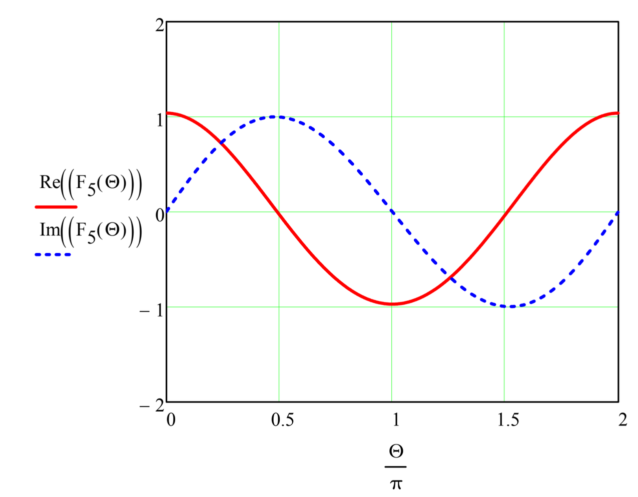
FIFTH ORDER

$$f_5(\theta) = -\int_0^{\theta}\int_0^{\theta}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3\,d\theta - i\frac{\pi^4}{90}\,d\theta + C_5 \qquad F_5(\Theta) = \sum_{n=1}^{\infty}\frac{e^{\Theta\cdot n\cdot i}}{n^5}$$

REAL SPACE



HARMONIC SPACE



$f_5(0) = 1.037 \qquad f_5\left(\frac{\pi}{2}\right) = -0.030 + 0.996i \qquad f_5(\pi) = -0.972$

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FIFTTH ORDER

Integrate the fourth order functions to obtain the fifth order:

Real Space

$$\int f_4(\theta)\,d\theta = \int\int_0^{\theta}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3\,d\theta - i\frac{\pi^4}{90}\,d\theta = \int_0^{\theta}\int_0^{\theta}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3\,d\theta - i\frac{\pi^4}{90}\,d\theta + C_5$$

with $C_5 = \frac{16}{31}\int_0^{\pi}\int_0^{\theta}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3\,d\theta - i\frac{\pi^4}{90}\,d\theta = \sum_{n}\frac{1}{n^5} = \mathrm{Li}_5(1)$

Harmonic Space

$$\int F_4(\Theta)\,d\Theta = \int i\sum_{n=1}^{\infty}\frac{e^{i\cdot n\cdot\Theta}}{n^4}\,d\Theta = \sum_{n=1}^{\infty}\frac{e^{\Theta\cdot n\cdot i}}{n^5}$$

Then the fifth order mapping relationship is $\underbrace{\int_0^{\theta}\int_0^{\theta}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3\,d\theta - i\frac{\pi^4}{90}\,d\theta + C_5}_{\text{Real Space}} = \underbrace{\sum_{n=1}^{\infty}\frac{e^{i\cdot n\cdot\Theta}}{n^5}}_{\text{Harmonic Space}}$

Evaluate at 0

$$-\int_0^{0}\int_0^{\theta}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3\,d\theta - i\frac{\pi^4}{90}\,d\theta + C_5 =$$

$$= \frac{e^{i\cdot n\cdot 0}}{n^5} = \begin{pmatrix}1.00000\\0.03125\\0.00412\\0.00098\\0.00032\\0.00013\\0.00000\\0.00000\end{pmatrix} = \sum_{n=1}^{\infty}\frac{1}{n^5} = \mathrm{Li}_5(1)$$

$$-\int_0^{0}\int_0^{\theta}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3\,d\theta - i\frac{\pi^4}{90}\,d\theta + C_5 = 1.037$$

But $\int_0^{0}\int_0^{\theta}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3\,d\theta - i\frac{\pi^4}{90}\,d\theta = 0$

So $\sum_{n=1}^{\infty}\frac{1}{n^5} = C_5 = \mathrm{Li}_5(1) = 1.037..$

Evaluate at $\frac{\pi}{2}$

$$-\int_0^{\frac{\pi}{2}}\int_0^{\theta}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3\,d\theta - i\frac{\pi^4}{90}\,d\theta + C_5 =$$

$$= \frac{e^{i\cdot n\cdot\frac{\pi}{2}}}{n^5} = \begin{pmatrix}1.00000i\\-0.03125\\-0.00412i\\0.00098\\0.00032i\\-0.00013\\0.00000\\0.00000\end{pmatrix} = i\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^5} - \frac{1}{32}\cdot\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^5} = \mathrm{Li}_5(i)$$

$$-\int_0^{\frac{\pi}{2}}\int_0^{\theta}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3\,d\theta - i\frac{\pi^4}{90}\,d\theta + C_5 = -0.030 + 0.996i$$

Only the Imaginary Component is needed, the Real is redundant:

$$\mathrm{Im}\int_0^{\frac{\pi}{2}}\int_0^{\theta}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3\,d\theta - i\frac{\pi^4}{90}\,d\theta + C_5$$

$$= i\frac{1}{32}\cdot\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta\,d\theta\,d\theta\,d\theta$$

So $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^5} = \frac{5}{1536}\cdot\pi^5 = i\frac{1}{32}\cdot\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta\,d\theta\,d\theta\,d\theta = \frac{\mathrm{Li}_5(i) - \mathrm{Li}_5(-i)}{2\cdot i} = 0.996...$

Evaluate at π

$$-\int_0^{\pi}\int_0^{\theta}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3\,d\theta - i\frac{\pi^4}{90}\,d\theta + C_5 =$$

$$\frac{e^{i\cdot n\cdot\pi}}{n^5} = \begin{pmatrix}-1.00000\\0.03125\\-0.00412\\0.00098\\-0.00032\\0.00013\\0.00000\\0.00000\end{pmatrix} = -\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^5} = \mathrm{Li}_5(-1)$$

$$-\int_0^{\pi}\int_0^{\theta}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3\,d\theta - i\frac{\pi^4}{90}\,d\theta + C_5 = -0.972$$

But $\int_0^{\pi}\int_0^{\theta}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3\,d\theta - i\frac{\pi^4}{90}\,d\theta + C_5 = \left(\frac{31}{16} - 1\right)\cdot C_5 = \frac{15}{16}\cdot C_5$

$$= \frac{15}{31}\cdot\int_0^{\pi}\int_0^{\theta}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta + i\cdot\frac{\pi^2}{6}\,d\theta + C_3\,d\theta - \frac{\pi^4}{90}\cdot i\,d\theta$$

So $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^5} = \frac{15}{16}\cdot C_5 = -\mathrm{Li}_5(-1) = 0.972...$

And $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5} = \frac{31}{32}\cdot\sum_{n=1}^{\infty}\frac{1}{n^5} = \frac{1}{2}\int_0^{\pi}\int_0^{\theta}\int_0^{\theta}\int_0^{\theta} \ln\left(1-e^{i\cdot\theta}\right)d\theta\,d\theta + C_3\,d\theta\,d\theta + \frac{\pi^5}{120}\cdot i$

So $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5} = \frac{31}{32}\cdot C_5 = \frac{31}{32}\mathrm{Li}_3(1) = 1.0045..$