Casey's Curves

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Methods of Determining Mapping equations for Higher Orders

Method of Products for Determining Graphical Equivalence at Higher Orders - This Method is Not Used

The fixed value for π is a first order relationship which can be determined by wave mechanics. Higher orders, like those involving π^2, can be determined with some degree of success using the product of the first order mapping function and a similar sine base wave function.

The first order Real Space and Harmonic Space functions $f_1(\theta)$ and $F_1(\Theta)$ were shown to be graphically equivalent for the range of interest, $-\pi$ to $+\pi$.

Our first order Harmonic Space mapping function $F_1(\Theta) = \sum_{n=1}^{\infty}\left[\frac{2}{n}\cdot\sin[n\cdot(\pi-\Theta)]\right]$ can equivalentally be written as $\sum_{n=1}^{\infty}\left[\frac{4}{n}\cdot\sin\left[n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\right]$ for this range. This is demonstrated on the graph below, and discussed on Sheet 1

Using a first order Harmonic Space mapping function $F_1(\Theta) = \sum_{n=1}^{\infty}\left[\frac{4}{n}\cdot\sin\left[n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\right]$, we found that that the product of this and multiples of $\frac{2}{n}\cdot\sin\left[n\cdot\left(\frac{\Theta}{2}\right)\right]$ could be used for graphical equaivalence at higher orders.

This simple multiplication of products based on the fist order mapping equation was used for investigating the first few orders of the series.

For the second order Harmonic Space mapping function we used $F_2(\Theta) = \sum_{n=1}^{\infty}\left[\left[\frac{4}{n}\cdot\sin\left[n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\right]\cdot\left[\frac{2}{n}\cdot\sin\left[n\cdot\left(\frac{\Theta}{2}\right)\right]\right]\right]$, with a general formula of $F_p(\Theta) = \sum_{n=1}^{\infty}\left[\left[\frac{4}{n}\cdot\sin\left[n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\right]\cdot\left[\frac{2}{n}\cdot\sin\left[n\cdot\left(\frac{\Theta}{2}\right)\right]\right]^{p-1}\right]$

where p is the order. And for the second order Real Space mapping function we used $f_2 = \theta^2$, with a general formula of $f_p(\theta) = \theta^p$.

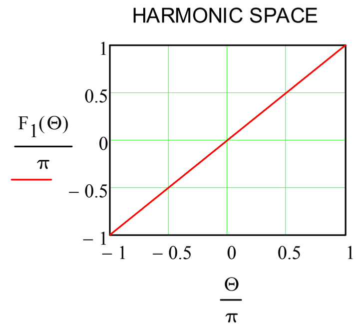
We mapped the Harmonic Space mapping functions next to the Real Space mapping functions for the first six orders below. It can be seen that these are graphically equivalent for the $-\pi$ to $+\pi$

for the first two orders only. Beyond this the range is limited. By the third order we are reduced to $-\frac{\pi}{2} < \Theta < +\frac{\pi}{2}$, or less .

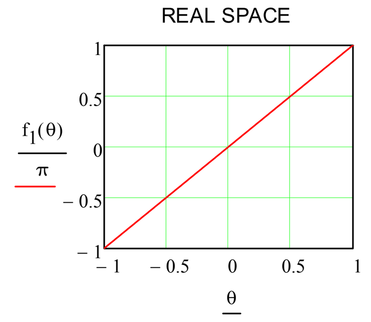
HARMONIC SPACE	REAL SPACE	HARMONIC SPACE	REAL SPACE
$F_1(\Theta) = \sum_{n=1}^{\infty}\left[\frac{4}{n}\cdot\sin\left[n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\right]$	$f_1(\theta) = \theta$	$F_2(\Theta) = \sum_{n=1}^{\infty}\left[\left[\frac{4}{n}\cdot\sin\left[n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\right]\cdot\left[\frac{2}{n}\cdot\sin\left[n\cdot\left(\frac{\Theta}{2}\right)\right]\right]\right]$	$f_2(\theta) := \theta^2$
$F_3(\Theta) = \sum_{n=1}^{\infty}\left[\left[\frac{4}{n}\cdot\sin\left[n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\right]\cdot\left[\frac{2}{n}\cdot\sin\left[n\cdot\left(\frac{\Theta}{2}\right)\right]\right]^2\right]$	$f_3(\theta) := \theta^3$	$F_4(\Theta) = \sum_{n=1}^{\infty}\left[\left[\frac{4}{n}\cdot\sin\left[n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\right]\cdot\left[\frac{2}{n}\cdot\sin\left[n\cdot\left(\frac{\Theta}{2}\right)\right]\right]^3\right]$	$f_4(\theta) := \theta^4$
$F_5(\Theta) = \sum_{n=1}^{\infty}\left[\left[\frac{4}{n}\cdot\sin\left[n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\right]\cdot\left[\frac{2}{n}\cdot\sin\left[n\cdot\left(\frac{\Theta}{2}\right)\right]\right]^4\right]$	$f_5(\theta) := \theta^5$	$F_6(\Theta) = \sum_{n=1}^{\infty}\left[\left[\frac{4}{n}\cdot\sin\left[n\cdot\left(\pi-\frac{\Theta}{2}\right)\right]\right]\cdot\left[\frac{2}{n}\cdot\sin\left[n\cdot\left(\frac{\Theta}{2}\right)\right]\right]^5\right]$	$f_6(\theta) := \theta^6$

FIRST ORDER

HARMONIC SPACE



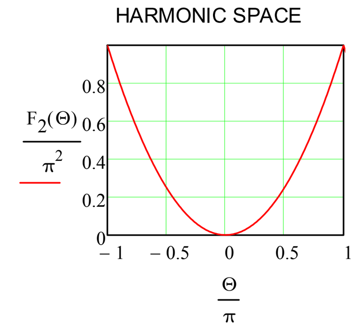
REAL SPACE



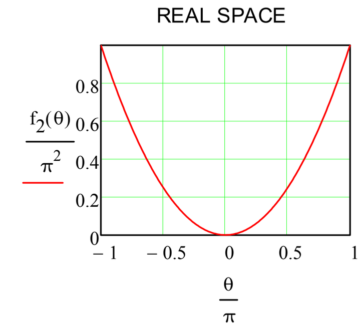
$\frac{F_1\left(\frac{\pi}{2}\right)}{\pi} = 0.5$ $\frac{f_1\left(\frac{\pi}{2}\right)}{\pi} = 0.5$

SECOND ORDER

HARMONIC SPACE



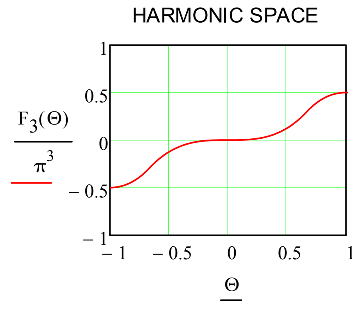
REAL SPACE



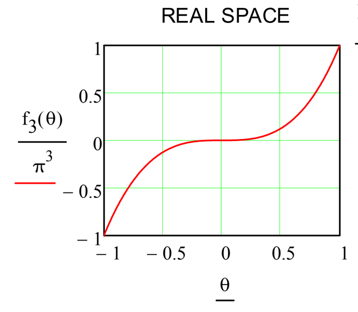
$\frac{F_2\left(\frac{\pi}{2}\right)}{\pi^2} = 0.25$ $\frac{f_2\left(\frac{\pi}{2}\right)}{\pi^2} = 0.25$

THIRD ORDER

HARMONIC SPACE



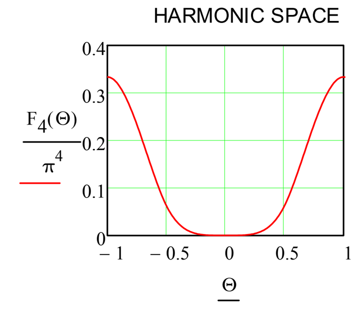
REAL SPACE



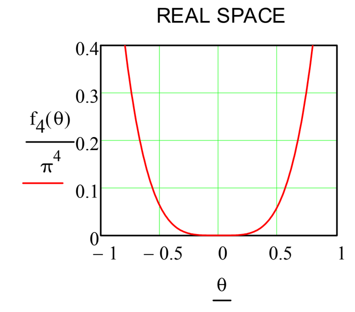
$\frac{F_3\left(\frac{\pi}{2}\right)}{\pi^3} = 0.125$ $\frac{f_3\left(\frac{\pi}{2}\right)}{\pi^3} = 0.125$

FOURTH ORDER

HARMONIC SPACE



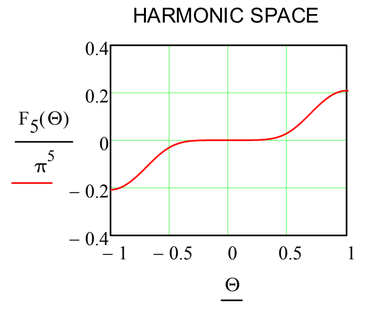
REAL SPACE



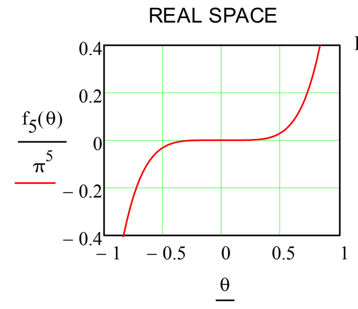
$\frac{F_4\left(\frac{\pi}{2}\right)}{\pi^4} = 0.0625$ $\frac{f_4\left(\frac{\pi}{2}\right)}{\pi^4} = 0.0625$

FIFTH ORDER

HARMONIC SPACE



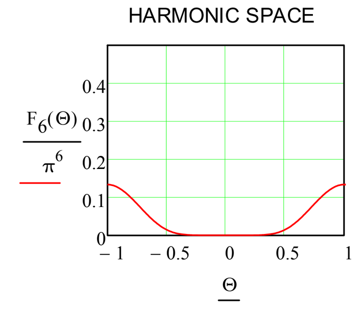
REAL SPACE



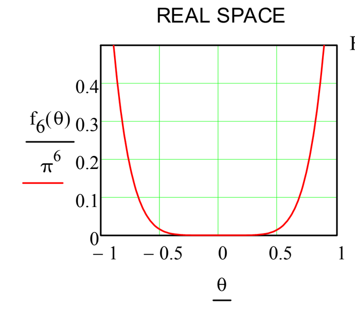
$F_5\left(\frac{\pi}{2}\right) = 9.3639$ $f_5\left(\frac{\pi}{2}\right) = 9.5631$

SIXTH ORDER

HARMONIC SPACE



REAL SPACE



$F_6\left(\frac{\pi}{2}\right) = 14.02$ $f_6\left(\frac{\pi}{2}\right) = 15.022$

While the equivalency relationships are correct for certain ranges, the range restriction at higher orders limits its usefulness.

Additionally, while the first order mapping function was based on wave mechanics and calculus, the product method is based primarily on graphical equivalence and is less rigid.

Because of these issues, the method of products is not used here but is included as a historic wrap up.

Method of Integration (Sheet 4.3)

Greater success was found using the method of integration. We build on the first order relationship determined by wave mechanics.

We used the first order mapping relationship $\theta = \sum_{n=1}^{\infty}\left[\frac{2}{n}\cdot\sin[n\cdot(\pi-\Theta)]\right]$ (Real Space = Harmonic Space). Since these are gradically equivalent, their integrals must be as well. If we integrate both sides:

$$\int \theta\, d\theta = \int \sum_{n=1}^{\infty}\left[\frac{2}{n}\cdot\sin[n\cdot(\pi-\Theta)]\right] d\Theta$$

(Real Space = Harmonic Space) If we integrate, detrminne the integration constant, and simplify, we obtain

$$\theta^2 = \sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cdot\cos[n\cdot(\pi-\Theta)]\right] + \frac{\pi^2}{3}$$

(Real Space = Harmonic Space)

This a π^2 relationship.Relationships with higher orders of π can be obtained by successive integration. On Sheet 4.3, we use this method to obtain the key values:

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{32}$	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4} = \frac{7}{720}\cdot\pi^4$	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^5} = \frac{5}{1536}\cdot\pi^5$
Alternating Odd	*Alternating All*	*Alternating Odd*	*Alternating All*	*Alternating Odd*

The use of the sine based set of wave functions, which is real, gives us a real result for all successive integrations. The series alternates between *alternating odd* and *alternating all* with ascending order. The series alternation is due to alternating between sine and cosine terms under integration - the sine term results in an *alternating odd* series and the cosine term results in an *alternating all* series. The geometric term is always a ratio of an order of π. The order of π also ascends due to n in the argument of the sine and cosine terms, which increases the exponent of n in the denominator under integration.

The results from this section are important in establishing closed values of for key parts of the table.

Integration of The Polylogarithm of the Euler Complex Exponential Function (Sheets 4.4 - 4.6)

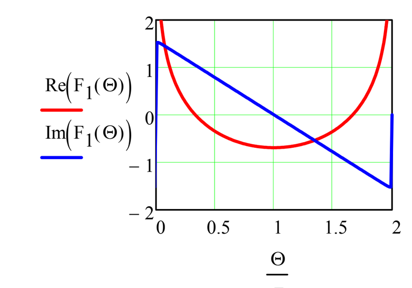
Key parts of the table have been established but much is missing due to the sine-cosine alternation.

We integrated a set of sine based wave functions to get the higher order mapping functions, and this was repsonsible for the alternation. The sine function in the first order Harmonic Space mapping function was the result of a definite integral creating the subtraction of two opposing spinners in the complex plane. The subtraction causes the imaginary component of every spinner to cancel out, yielding a set of real, sine-based, results. The cancellation process results in the loss of one of the spinner components.This lost component is needed to generate the missing series in the table that is lost in the alternation.

If use a indefinite integral on the spinner set instead of an definite integral, the subtraction of the two spinners won't occur, and both the real and imaginary components will be preserved. In this way we can get relationships for the missing series.

We use as a basis space $\sum_{n=1}^{\infty} e^{i\cdot n\cdot\Theta}$ which is our set of unit spinners with continuity requirement nλ=2π. $\int e^{i\cdot n\cdot\Theta}\, d\Theta = \frac{e^{i\cdot n\cdot\Theta}}{n} + C$.

Then the first order Harmonic Space mapping function will be $F_1(\Theta) = \sum_{n=1}^{\infty}\frac{e^{i\cdot n\cdot\Theta}}{n}$. This is our familiar set of n spinners, each with amplitude 1/n.



HARMONIC SPACE

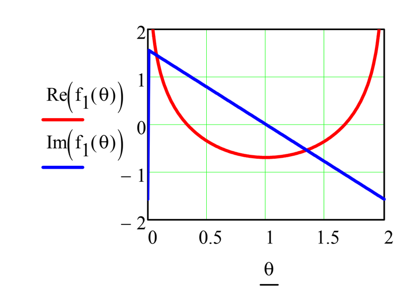
$F_1(\Theta) = \sum_{n=1}^{\infty}\frac{e^{i\cdot n\cdot\Theta}}{n}$

It has both real and imaginary terms, and will integrate successively to $F_p(\Theta) = \sum_{n=1}^{\infty}\frac{e^{i\cdot n\cdot\Theta}}{n^p} + C$

This can also be identified as the polylogarithm $Li_p(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^p}$ with argument $z = e^{i\cdot n\cdot\Theta}$, a quantized form of the Euler complex exponential function.

At the quadrants $e^{i\cdot\Theta}$ has values +1, +i, -1 & -i, which are used in the infinite series. Thus our evaluation points would typically use Θ = 0, π/2, π...

The Real Space mapping Function is $f_1(\theta) = -\ln\left(1-e^{1i\theta}\right)$.



REAL SPACE

$f_1(\theta) = -\ln\left(1-e^{1i\theta}\right)$

It also has both real and imaginary components and is graphically equivalent to the Harmonic Space map. We examined this on Sheet 3.2.

This can also be identified with the polylog, which has an alternate definition $Li_1(z) = -\ln(1-z)$ for p = 1, and $Li_{p+1}(z) = \int_0^z \frac{Li_p(t)}{t}\,dt$

for all other p. The polylog has with both harmonic and real space definitions.

Because of the graphical equivalence of the first order maps, successive integrals will be graphically equivalent. This is performed on Sheets 4.4 - 4.6.

The missing series are seen to be logarithms, polylogarithms, and ratios of π^p. Some of the key values are:

Alternating All	*Alternating Odd*	*Alternating All*	*Alternating Odd*	*Alternating All*
$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = \ln(2)$	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2} = \frac{Li_2(i)-Li_2(-i)}{2\cdot i}$	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3} = -Li_3(-1)$	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^4} = \frac{Li_4(i)-Li_4(-i)}{2\cdot i}$	$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^5} = -Li_5(-1)$

$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$	$\sum_{n=1}^{\infty}\frac{1}{n^3} = Li_3(1)$	$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$	$\sum_{n=1}^{\infty}\frac{1}{n^5} = Li_5(1)$
All	*All*	*All*	*All*

The results from this section provide closed values of for the remaining series. Taken combined, all of the series of the table are known or can be algebraically determmined.